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Self-Organized Criticality and Thermodynamic Formalism

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We develop a thermodynamic formalism for a dissipative version of the Zhang model of Self-Organized Criticality, where a parameter allows us to tune the local energy dissipation. By constructing a suitable Markov partition we define Gibbs measures (in the sense of Sinai, Ruelle, and Bowen), partition functions, and topological pressure allowing the analysis of probability distributions of avalanches. We discuss the infinite-size limit in this setting. In particular, we show that a Lee–Yang phenomenon occurs in the conservative case. This suggests new connections to classical critical phenomena.

KEY WORDS: Self-organized criticality; hyperbolic dynamical systems with singularities; thermodynamic formalism; Lee–Yang singularity.

In 1988, for the first time, Bak, Tang, and Wiesenfeld (BTW)[(1)] proposed a mechanism allowing a dynamical system to reach "spontaneously" a steady state exhibiting some analogies with a thermodynamic system at a critical point. In the BTW paradigm an external perturbation may induce a chain reaction or *avalanche* in the system. Then, when submitted to a stationary and adiabatic flux of external perturbations, the system is driven to a stationary state where avalanches are distributed according to a truncated *power law*, the cutoff being due to finite-size effects. This phenomenon, called by these authors Self-Organized Criticality (SOC), was quite unexpected, since attaining the critical state of a thermodynamic system usually requires a fine tuning of some control parameter (temperature, magnetic field, etc.), which is absent from the definition of the BTW model or of its many variants.[(2, 3)] This big difference between SOC systems and thermodynamic systems which

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exhibit a second-order phase transition was certainly the main point which attracted the physics community. There are, however, other differences certainly explaining why, despite the huge number of papers written on the subject, there is still no general scheme for analyzing these systems properly.

The analysis of a critical regime requires first a precise characterization of the steady state in a *finite-size* system. In equilibrium statistical mechanics the starting point is the definition of an ensemble (microcanonical, canonical, grand canonical,...) corresponding to a given set of constraints (e.g., quantities conserved, on average, by the microscopic dynamics, such as the energy or the number of particles). Then, the variational principle, stating that the entropy is maximum at equilibrium, defines, for finite-size and given constraints, a unique probability measure on the phase space: the Gibbs distribution. The free-energy density is the generating function of the cumulants. In the next step, one has to define properly the thermodynamic limit. This can be done via the Dobrushin–Landford–Ruelle (DLR) construction. Under suitable conditions on the interactions one can show that the infinite volume Gibbs measure exists as well as the limit of the free-energy density.[5] Away from phase transitions the free-energy density is, moreover, analytic and is used to compute the ensemble averages in the thermodynamic limit. On the other hand, at a phase-transition point, the free-energy density loses analyticity. A fruitful analysis of the corresponding singularity can be performed from the behavior of the zeros of the finite-size partition function ("Lee–Yang" zeros) when the size increases.[6–12]

The usual methods of statistical mechanics are not directly applicable to SOC systems. The stationary regime being the result of a specific non-Hamiltonian microscopic dynamics, there is no "natural" Gibbs distribution nor free energy in SOC systems. Moreover, the steady state corresponds to a nonequilibrium situation since the flux of externally injected energy is dissipated, on average, in the bulk or at the boundaries, inducing a constant energy flux through the system. But the presence of thresholds in the definition of the dynamics implies that the energy can be accumulated locally, eventually generating a chain reaction which may transport the energy on arbitrary large scales. Consequently, it is not possible to analyze the out-of-equilibrium stationary state via a local equilibrium hypothesis where one decomposes the system into mesoscopic cells locally at equilibrium. Finally, most of the results in this field rely on numerical simulations of finite-size systems, and the properties of the infinite-volume system have to be extrapolated from these data. Unfortunately, to handle these numerical results properly it is imperative to know the right finite-size scaling form and to have good control of the *bias* induced by numerics. Since there is no general theory, different Ansatz have been proposed,[13–15] but there are still debates on the correct finite-size scaling.

In refs. 16–19, we proposed an analysis of one SOC model, the Zhang model,[20] using tools and concepts from dynamical system theory and ergodic theory. We realized that the Zhang model is one of the few dynamical systems where one can establish a connection from the microscopic dynamics to a macroscopic description by using standard tools from dynamical system theory. In particular, our results opened up the possibility of constructing a thermodynamic formalism in the sense of Sinai, Ruelle, and Bowen.[21–23] In this setting, one defines in particular an extension of the notion of Gibbs measure, where the Hamiltonian is replaced by a dynamically relevant quantity. Like the "standard" Gibbs distribution, these measures are constructed from a variational principle (see the Appendix). We discussed in refs. 16 and 18 the possibility of using this formalism to relate the microscopic dynamics to some macroscopic characteristics of the critical state. On the other hand, we argued that in this setting one may properly define the stationary state of the finite-size dynamical system, thereby allowing for a correct extrapolation of its properties when the size tends to infinity. This opened up, therefore, the possibility of solving some of the difficulties discussed above and of bringing a new contribution to the analysis of SOC models.

In the present paper we develop this aspect. More precisely, we show how the thermodynamic formalism may be used in the Zhang model to construct the equivalent of finite-volume Gibbs measures where the Hamiltonian is replaced by a dynamically relevant potential. This allows us to obtain generating functions of the avalanche distributions, which are the formal equivalent to partition functions and free energy in statistical mechanics. We then develop a method inspired by the Lee–Yang zeros analysis in the usual critical phenomena. This allows us to characterize the SOC state by extrapolating to the infinite-size limit and to gain a theoretical control on the validity of the extrapolations made from numerical simulations (this last aspect is fully developed in ref. 24).

For this purpose, we introduce a nonconservative version of Zhang model, where a parameter h controls the local energy dissipation. This parameter, as we show, has to be tuned to $h = 0$ to obtain a critical state in the thermodynamic limit. Consequently, our model "self-organizes" into a "critical" state only in the conservative case. However, the presence of this additional parameter gives us some freedom to control the dynamics. Moreover, it allows us to outline the role of boundary dissipation in the process of self-organization into a critical state.

The paper is structured as follows. The first part is devoted to the analysis of the dynamical properties of the finite-size model. After a presentation of the model and of the general setting (Sections 1.1 and 1.2), we discuss some fundamental properties of the stationary state in a finite-size system (Section 1.3). We show that the finite-size system is weakly hyperbolic (all Lyapunov exponents bounded away from zero). We also establish a link between the energy transport and the Lyapunov exponents that play a role similar to the eigenvalues of the Laplace operator in diffusion (Section 1.3.2). We then derive explicit equations for the first negative Lyapunov exponent, the density of active sites, and the dissipation rate (Section 1.3.3). This allows us to make an extrapolation of the model properties when $L \to \infty$ (Sections 1.3.4 and 1.3.5).

The second part develops the thermodynamic formalism and shows, in particular, that the full dynamics can be described by a finite Markov chain (sub-shift of finite type) while the initial dynamical system has uncountably many states. The construction is based on a redefinition of the dynamical system in terms of return maps that play a role similar to the toppling operators in the Abelian sandpile.[25] More precisely, there is a one-to-one correspondence between each return map and each avalanche (Section 2.1). In Section 2.2, we show that almost every point has local stable and unstable manifolds. This result is partially based on a rigorous statement and partially on a numerical simulation. Then we construct in Section 2.3 a Markov partition allowing a symbolic coding of the dynamics in which each symbol corresponds to a given avalanche. We discuss in Section 2.4 the properties of the corresponding Markov chain. We propose in Section 2.5 several versions of Gibbs measures from which we can extract equations such as Eqs. (48), (49), (50), and (65) linking some characteristics of the microscopic dynamics to the avalanche distributions. We discuss the finite-size scaling (Section 2.6) and the thermodynamic limit in this setting. It is then shown, in Section 2.7, that when the size of the system diverges, a *Lee–Yang phenomenon*[6] occurs, related to a loss of analyticity of the topological pressure. This unexpected result opens up an effective way to "map self-organized criticality to criticality."[26] We show that the Lee–Yang phenomenon is observed only when the energy is *locally conserved* ($h = 0$). Moreover, the critical exponent characterizing the divergence of the correlation length of the avalanche size distribution when $h \to 0$ is analytically related to the angle that the Lee–Yang zeros form with the real axis, in strong analogy with the usual critical phenomena. The Appendix presents a short presentation of the thermodynamic formalism, for the benefit of non-specialists.

1. GENERAL SETTING

1.1. Definitions

The Zhang model is defined as follows. Let Λ be a d-dimensional box in $\mathbb{Z}^d$, taken as a square of edge length L, for simplicity. Denote

$N = \#\Lambda = L^d$, where # denotes the cardinality of a set. Each site $i \in \Lambda$ is characterized by its "energy" X_i, which is a nonnegative real and finite number. Denote by $\mathbf{X} = \{X_i\}_{i \in \Lambda}$ a configuration of energies. Let E_c be a real, positive number, called the *critical energy*, and $\mathcal{M} = [0, E_c[^N$. A configuration $\mathbf{X}$ is *stable* when $\mathbf{X} \in \mathcal{M}$ and *unstable* otherwise. In an unstable configuration, the sites i such that $X_i \geqslant E_c$ are called *active* or unstable. The dynamics on $\mathbf{X}$ depends on whether $\mathbf{X}$ is stable or unstable.

If $\mathbf{X}$ is stable, one chooses a site $i \in \Lambda$ at random with probability $\frac{1}{N}$ and adds to it the energy $\delta = 1$ (*excitation*). If $\mathbf{X}$ is unstable, each active site loses a part of its energy, redistributed in equal parts to its $2d$ neighbors in the following way (*relaxation*). Fix two[3] real parameters, $\epsilon \in [0, 1[$ and $h \geqslant 0$. Set $\gamma = \epsilon^h$, $\epsilon' = \epsilon\gamma = \epsilon^{1+h}$, and $\alpha = \frac{(1-\epsilon)}{2d}$. When i is active it gives the energy αX_i to its $2d$ neighbors and keeps the energy $\epsilon' X_i$. Therefore the energy is locally conserved in the case $h = 0$, whereas there is local dissipation[4] when $h > 0$. The relaxation dynamics is synchronous, and it is useful to express it in terms of the map

$$\mathbf{F}(\mathbf{X}) = \mathbf{X} + (\gamma - 1)\, \epsilon \mathbf{Z}(\mathbf{X}) * \mathbf{X} + \alpha \Delta[\mathbf{Z}(\mathbf{X}) * \mathbf{X}]. \tag{1}$$

In this equation, $\mathbf{Z}(\mathbf{X})$ is an N-dimensional vector, such that $Z_i(\mathbf{X}) = 0$ if $X_i < E_c$ and $Z_i(\mathbf{X}) = 1$ if $X_i \geqslant E_c$. The $*$ denotes the product component by component: if $\mathbf{X}$ and $\mathbf{Y}$ are N-dimensional vectors, $\mathbf{X} * \mathbf{Y}$ is the N-dimensional vector of components $X_i Y_i$. Δ is the discrete Laplacian on Λ.

The succession of relaxations leading an unstable configuration to a stable one is called an *avalanche*. Let $^\partial\Lambda$ be the boundary of Λ, namely the set of points in $\mathbb{Z}^d \setminus \Lambda$ at a distance 1 from Λ. The sites of $^\partial\Lambda$ always have zero energy (dissipation at the boundaries). Since the total energy in a finite lattice is finite, all avalanches stop within a *finite* number of iterations for $h \geqslant 0$. The addition of energy is *adiabatic*. When an avalanche occurs, one waits until it stops before adding a new energy quantum. Further excitations eventually generate a new avalanche, but, because of the adiabatic rule, each new avalanche starts from *only one* active site.

The structure of an avalanche is encoded by the sequence of active sites:

$$\mathcal{C} = \{C_t\}_{1 \leqslant t \leqslant \tau_{\mathcal{C}}}, \tag{2}$$

[3] Note that the original Zhang model corresponds to the case $\epsilon = 0$. The straightforward extension proposed here allows us to avoid pathological dynamical effects due to the existence of zero eigenvalues for the tangent maps when $\epsilon = 0$.[18]

[4] The case $h < 0$ would correspond to local energy injection. However, in this case there may not exist a stationary regime (see Section 1.3.3).

where

$$C_t = \{ j \in \Lambda \mid X_j \geqslant E_c \text{ in the } t\text{th step of the avalanche} \} \tag{3}$$

and where $\tau_{\mathscr{C}}$, called the *avalanche duration*, is the smallest number such that $C_{\tau_{\mathscr{C}}+1} = \varnothing$. Each avalanche can be labeled by a double index (i, j). The first index refers to the site where the energy is dropped and the second index labels the different avalanches starting at i (including the "empty" avalanche, where the excitation of i does not render it active). Moreover, to each avalanche (i, j) corresponds a convex domain $\mathscr{M}_{(i,j)}$ in $\mathscr{M}$. For a fixed i, these domains form a partition of $\mathscr{M}$.[18]

The total energy of a stable configuration in a finite lattice being finite (it is bounded by $L^d E_c$), the total number of different avalanches is finite for finite L (but diverges as $L \to \infty$ when $h = 0$ as discussed below). Call $l(i) < \infty$ the number of different avalanches starting at i (note that $l(i)$ depends on L and h). The *size s* of an avalanche is the total number of active sites, $s(\mathscr{C}) = \# \bigcup_t C_t$. The *area a* is the number of distinct active sites in $\mathscr{C}$. We will generically denote by n an avalanche observable (size, duration, area) and $n(i, j)$ will be the value that n takes in the avalanche (i, j).

Because all avalanches are *finite* (for finite L), and since we are not interested in the transients, one can, without loss of generality, take all initial energy configurations $\mathbf{X} \in \mathscr{M}$. All trajectories starting from $\mathscr{M}$ belong to a compact set $\mathscr{D}$. Denote $\bar{\mathscr{M}} = \mathscr{D} \setminus \mathscr{M}$. $\bar{\mathscr{M}}$ contains the set of all unstable energy configurations achievable in an avalanche starting from an energy configuration in $\mathscr{M}$. It is useful to encode the dynamics of excitation in the following way. Let Σ_Λ^+ be the set of right infinite sequences $\tilde{a} = \{a_1,..., a_k,...\}$, $a_k \in \Lambda$, and σ be the *left shift*[5] over Σ_Λ^+, namely $\sigma\tilde{a} = a_2 a_3 \cdots$. The elements of Σ_Λ^+ are called *excitation sequences*. The set $\Sigma_\Lambda^+ \times \mathscr{D}$ is the *phase space* of the Zhang model and $\hat{\mathbf{X}} = (\tilde{a}, \mathbf{X})$ is a point in the phase space. The Zhang model dynamics is given by a map $\hat{\mathbf{F}}: \Sigma_\Lambda^+ \times \mathscr{D} \to \Sigma_\Lambda^+ \times \mathscr{D}$ such that:

$$\mathbf{X} \in \mathscr{M} \Rightarrow \hat{\mathbf{F}}(\hat{\mathbf{X}}) = (\sigma\tilde{a}, \mathbf{X} + \mathbf{e}_{a_1}) \qquad \text{(Excitation)}, \tag{4}$$

$$\mathbf{X} \in \bar{\mathscr{M}} \Rightarrow \hat{\mathbf{F}}(\hat{\mathbf{X}}) = (\tilde{a}, \mathbf{F}(\mathbf{X})) \qquad \text{(Relaxation)}, \tag{5}$$

where $\mathbf{e}_a$ is the ath canonical basis vector of $\mathbb{R}^N$. The knowledge of an initial energy configuration $\mathbf{X}$ and of an excitation sequence $\tilde{a}$ fully determines the evolution. One can endow Σ_Λ^+ with a probability distribution

[5] Throughout this paper, we will often think of the left Bernoulli shift on Σ_Λ^+ as represented by the system $z \to Nz \bmod 1$, $z \in [0, 1]$. In particular, this allows for a definition of the tangent map in the direction of the shift.

corresponding to a random choice of the excited sites. The excited sites are chosen at random and independently with uniform probability. This corresponds to endowing Σ_Λ^+ with the *uniform Bernoulli measure*, denoted by ν_L in the sequel.

1.2. The Two SOC Conjectures

The model definition entails the convergence of the dynamics to a stationary state where the average incoming energy flux is compensated by the dissipated energy flux. It is furthermore *conjectured* in the SOC literature that the model is "ergodic." Although this terminology is common in the physics literature, it is misleading since the Zhang model admits *infinitely many* ergodic measures. In particular, one may generate infinitely many different stationary states by periodic excitation (in this case, one can easily show that the corresponding ergodic measure has its support on a limit cycle). What is implicitly meant is that the stationary state is reached for *generic* choices of the initial energy configuration and of the excitation sequence. In the dynamical systems terminology, this corresponds to assuming that there exists a *unique natural* or *Sinai–Ruelle–Bowen* (SRB) measure.[21–23] [6] On physical grounds, the existence of a unique SRB measure is the minimal required property since it implies that, if one selects the initial conditions at random with a uniform probability (or any probability possessing a density), then the time average will not depend on the initial condition.

Although implicitly assumed in almost all SOC papers, this property is far from being trivially true. (Actually, there are only a few physical models

[6] Let $\hat{\mu}_L$ be an invariant measure on $\Sigma_\Lambda^+ \times \mathscr{D}$. Let ψ be some function, integrable with respect to $\hat{\mu}_L$. Denote by $\bar{\psi}_L$ the time average:

$$\bar{\psi}_L(\hat{\mathbf{X}}) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \psi(\hat{\mathbf{F}}^t(\hat{\mathbf{X}})), \tag{6}$$

and let $\int \psi(\hat{\mathbf{X}})\, d\hat{\mu}_L(\hat{\mathbf{X}})$ be the ensemble average with respect to $\hat{\mu}_L$. Then $\hat{\mu}_L$ is an SRB measure if $\bar{\psi}_L(\hat{\mathbf{X}}) = \int \psi(\hat{\mathbf{X}})\, d\hat{\mu}_L(\hat{\mathbf{X}})$ for a set of initial conditions $\hat{\mathbf{X}}$ of *positive Lebesgue measure.*[27] If the SRB measure is unique, then the time average and the ensemble average with respect to $\hat{\mu}_L$ are equal for a set of initial conditions of *full* Lebesgue measure. Moreover, if $\hat{\mathbf{F}}$ is topologically mixing, namely if for any open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \mathscr{A}$, $\exists T \equiv T(\mathcal{O}_1, \mathcal{O}_2) > 0$ such that $\mathcal{O}_1 \cap \hat{\mathbf{F}}^t \mathcal{O}_2 \neq \varnothing$, $\forall t \geqslant T$, then the SRB measure is the weak limit:

$$\hat{\mu}_L = \lim_{t\to\infty} \hat{\mathbf{F}}^t(\nu_L \times \mu_L^{(0)}) \tag{7}$$

where ν_L is the excitation measure, $\mu_L^{(0)}$ the Lebesgue measure on the space of energy configurations, and $\hat{\mathbf{F}}^{*t}(\nu_L \times \mu_L^{(0)})$ is the image of $\nu_L \times \mu_L^{(0)}$ under the map $\hat{\mathbf{F}}^t$.

where the existence and uniqueness of an SRB measure is rigorously established.) This has been widely discussed in a previous paper,[18] where strong mathematical arguments in favor of this were given. Actually, the existence of a unique SRB measure was proved but restricted to the one-dimensional model and to some E_c interval. We will go one step further in Section 2.2 but will not completely prove this property in this paper. Consequently, we will state it as a conjecture in the sequel, the *first SOC conjecture*:

Conjecture 1. For any finite L, there exists a unique SRB measure $\hat{\mu}_L$ for generic values of the parameters $E_c, \epsilon, h \geqslant 0$ and whatever the lattice dimension $d < \infty$.

Call $P_L(n, h)$ the probability distribution[7] of the avalanche observable n in the stationary regime for a lattice of size L and denote by $\langle n \rangle_{L,h}$ the corresponding average

$$\langle n \rangle_{L,h} = \sum_{n=0}^{\xi_L^n(h)} n P_L(n, h), \tag{8}$$

where $\xi_L^n(h)$ is the maximal value that the random variable n can take, among all the avalanches having a nonzero probability of occurrence in the stationary regime, in a lattice of size L. For simplicity, we will write $P_L(n, 0) = P_L(n)$ $(\langle n \rangle_{L,0} = \langle n \rangle_L)$ for the conservative case $h = 0$. The numerical simulations report that $P_L(n)$ has a power-law shape over a finite range, with a cutoff corresponding to finite-size effects. As L increases, the power-law range increases. In the SOC literature it is assumed that the limit $L \to \infty$ of the model is well defined, and that the corresponding probability distribution for n, $P^*(n)$, is a power law. This is again a conjecture that has not yet been proved in the Zhang model. Let us call it the *second SOC conjecture*:

Conjecture 2. As $L \to \infty$, for $h = 0$, and for each observable n, $P_L(n)$ converges to a power law

$$P^*(n) = \frac{K}{n^{\tau_n}}, \qquad n = n_0 > 0 \cdots \infty.$$

This entails the scale invariance of the corresponding stationary state, which has, therefore, some common features with a critical state. τ_n is called the *critical exponent* of the observable n. It is commonly admitted

[7] By definition this is the $\hat{\mu}_L$ measure of the union of domains $\mathcal{M}_{i,j}$ such that $n(i, j) = n$.

in the SOC community that a classification of the models can be made through the knowledge of their critical exponents ("universality classes").

We will not prove this conjecture but will give new insights based partially on the rigorous formalism developed in this paper and partially on numerical results. Before this we discuss in the next sections some fundamental properties of the Zhang model.

1.3. Characterizations of the SRB State

1.3.1. Macroscopic Observables

Since the SRB measure is the stationary state, the physically relevant quantities characterizing the stationary regime are obtained from $\hat{\mu}_L$. Denote $\bar{\omega}_L \stackrel{\text{def}}{=} \hat{\mu}_L(\Sigma_\Lambda^+ \times \mathscr{M})$. By definition this is the probability of injecting energy in the system at a given time or *excitation rate*. The average incoming energy flux, at stationarity, is given by the vector $\bar{\Phi}_L$ with components:

$$\bar{\Phi}_L(i) = \frac{\bar{\omega}_L}{N} \stackrel{\text{def}}{=} \bar{\phi}_L, \tag{9}$$

corresponding to the average energy received by the site i per unit time. Note that in this paper we consider, for simplicity, the case where the energy is uniformly distributed in the lattice, though most of the results hold in the more general case where $\bar{\Phi}_L$ is not spatially uniform.

Define

$$\rho_L(i) = \text{Prob}[X_i(t) \geqslant E_c] \stackrel{\text{def}}{=} \hat{\mu}_L[\{\hat{\mathbf{X}}, X_i \geqslant E_c\}], \tag{10}$$

the probability that the site i is active in the stationary regime, and let ρ_L be an N-dimensional vector, $\rho_L = \{\rho_L(i)\}_{i=1}^N$. The quantity

$$\rho_L^{\text{av}} = \frac{1}{N} \sum_{i=1}^{N} \rho_L(i) \tag{11}$$

is often called the *density of active sites* in the literature. It has been introduced by Vespignani and Zapperi[29] as an order parameter in SOC models. Moreover, these authors introduced the quantity

$$\chi_L \stackrel{\text{def}}{=} \frac{\partial \rho_L^{\text{av}}}{\partial \bar{\phi}_L}, \tag{12}$$

which they interpret as a *response function* with respect to variations in the excitation rate. In particular, considering the excitation at a given point and at a given time as a perturbation in the relaxation dynamics, χ_L characterizes the linear response to this perturbation. Note that χ_L depends on h. In particular, we show below that, in our model, χ_L *diverges* when $h=0$ and $L \to \infty$, as expected for a critical phenomenon.

In the following subsections, we show that the finite Zhang model is weakly hyperbolic (all Lyapunov exponents bounded away from zero). This is an important step in our construction of the thermodynamic formalism. We have, in fact, a deeper result. On the basis of the theory developed in ref. 19, one may construct an operator, called the transport operator, whose eigenvalues correspond to the less negative Lyapunov exponents (corresponding henceforth to the longest time scales). This allows us to obtain an explicit formula for the first negative Lyapunov exponent, the density of the active site, and the energy dissipation rate. Then, an extrapolation to the limit $L \to \infty$ shows that *the Zhang model loses hyperbolicity, in the conservative case, when* $L \to \infty$ (but remains hyperbolic for $h > 0$). The hyperbolicity implies that the finite-size Zhang model has exponential time-correlation decay. In the opposite case, the loss of hyperbolicity in the limit $L \to \infty$ corresponds, in the Zhang model, to a divergence of the correlation-decay rate and the time-correlation decay becomes algebraic in the thermodynamic limit, as expected for a critical phenomenon.

1.3.2. Hyperbolicity, Lyapunov Exponents, and Energy Transport

Let $D\hat{\mathbf{F}}^t_{\hat{\mathbf{X}}}$ be the derivative[8] of $\hat{\mathbf{F}}^t$ at $\hat{\mathbf{X}}$. The particular structure of the dynamics (4) implies that, for all t, $D\hat{\mathbf{F}}^t_{\hat{\mathbf{X}}}$ is block diagonal. The one-dimensional block corresponds to a positive Lyapunov exponent:

$$\lambda_L(0) = \bar{\omega}_L \log(N). \tag{13}$$

The excitation dynamics is therefore *expansive* with a rate proportional to the excitation rate $\bar{\omega}_L$. Note that $\lambda_L(0)$ is also the Kolmogorov–Sinai (KS) entropy. The relaxation dynamics being contracting,[18] the N corresponding Lyapunov exponents are *strictly* negative. Order them so that $0 > \lambda_L(1) \geqslant \lambda_L(2) \geqslant \cdots \geqslant \lambda_L(N)$ and denote by $\mathbf{v}_i(\hat{\mathbf{X}})$ in $\mathbb{R}^N$ the *Oseledec mode* corresponding to $\lambda_L(i)$, for the trajectory $\hat{\mathbf{X}}$.

It has been shown in ref. 19 that the negative Lyapunov exponents corresponding to slow time[9] scales can be computed from a linear operator $\mathscr{L}: \mathbb{R}^N \to \mathbb{R}^N$ which acts on a vector $\mathbf{v} \in \mathbb{R}^N$ as:

$$\mathscr{L}(\mathbf{v}) = \mathbf{v} + 2(\gamma - 1)\, \epsilon \rho_L * \mathbf{v} + 2\alpha \Delta(\rho_L * \mathbf{v}). \tag{14}$$

[8] Note that this matrix is piecewise constant with $\hat{\mathbf{X}}$ (it has jumps at every point $\hat{\mathbf{X}}$ such that $\exists i$ with $X_i = E_c$).

[9] Namely, time scales quite a bit longer than the exponential rate of correlation decay.

$\mathscr{L}$ has the structure of the transition operator in a diffusion process. It is obtained by associating to the relaxation dynamics the motion of a tagged particle. The particle stays at the same place if the corresponding site i is stable ($Z_i(\mathbf{X}) = 0$). If X_i is unstable ($Z_i(\mathbf{X}) = 1$) the particle has a (conditional) probability α of jumping to a neighbor site, a probability ϵ' of staying at the same place, and a probability $\epsilon - \epsilon' = (1-\gamma)\,\epsilon$ of disappearing (dissipation in the bulk). Using a Markovian approximation, valid only for time scales quite bit larger than the correlation decay, and taking into account some additional constraint in the dynamics (a site cannot relax in two successive time steps), one shows that the transition operator of this process is $\mathscr{L}$.[19] We call it the *transport operator* in the sequel.

The slow Lyapunov exponents are well approximated by the eigenvalues of $\mathscr{L}$. We have drawn in Fig. 1 the Lyapunov spectrum for $L = 20$, $E_c = 2.2$, $\epsilon = 0.1$, $h = 0.1$, $d = 2$ and the eigenvalues of $\mathscr{L}$, with (Fig. 1(a)) and without (Fig. 1(b)) boundaries (in this case Λ is a 2 dimensional torus). The results show that the slowest modes are well approximated while there is an increasing discrepancy when the characteristic time decreases. This corresponds to the failure of the Markovian approximation used to compute $\mathscr{L}$. The slow eigenmodes of $\mathscr{L}$ correspond to the Oseledec modes of the relaxation dynamics and are the analogs of Fourier modes in normal diffusion.

The largest negative Lyapunov, $\lambda_L(1)$, defines a characteristic time scale or *escape rate*, $t_L(1) = \frac{1}{|\lambda_L(1)|}$, corresponding to the time taken by a particle to exit the system, either by disappearing in the bulk or by escaping

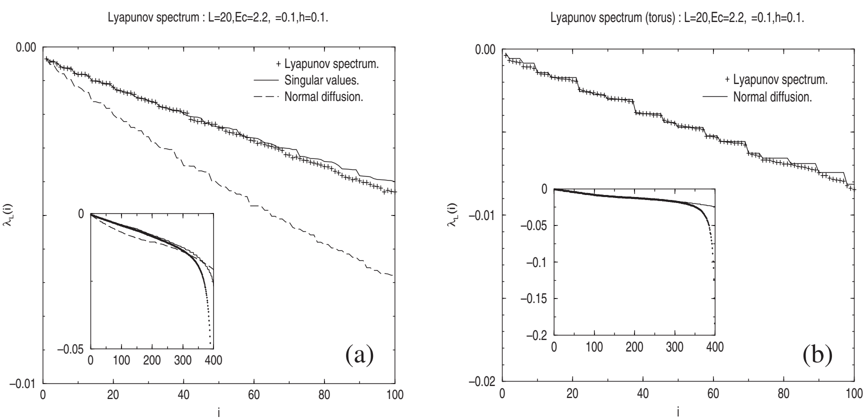

Fig. 1. (a) Lyapunov spectrum for $L = 20$, $E_c = 2.2$, $\epsilon = 0.1$, $h = 0.1$, $d = 2$ and the singular values of (14). (b) The Lyapunov spectrum for the same parameters values but without boundaries (torus).

from the boundaries. In terms of the dynamical system (4) the largest negative Lyapunov exponent is the exponential decay rate of a generic perturbation around some point $\mathbf{X}$. Consequently, $\lambda_L(1)$ also gives the decay rate of the initial local perturbation induced by the excitation and $t_L(1)$ defines the characteristic time required for this perturbation to vanish. The first Lyapunov exponent is well approximated by (Fig. 2):

$$\lambda_L(1) \sim -\frac{2\bar{\omega}_L}{E^+_{\text{tot}}}, \tag{15}$$

where $E^+_{\text{tot}} = \sum_{i=1}^N \bar{X}^+_L(i)$ and $\bar{X}^+_L(i)$ is the *conditional expectation* of X_i given that $X_i \geqslant E_c$ (see next section). Therefore $\lambda_L(1)$ scales like the dissipated energy per unit time (see Theorem 2 in ref. 19).

In the presence of boundaries ρ_L is nonuniform in the lattice (see Fig. 3(a)). The consequence is that the Lyapunov spectrum departs from the spectrum of a normal diffusion that one would obtain if ρ_L were to be spatially uniform (except for the first eigenvalue). In particular, at time scales of the order of the average duration of an avalanche, one obtains an anomalous diffusion exponent z directly related to the scaling properties of the Lyapunov spectrum.[19] On the other hand, were ρ_L to be uniform in the lattice ($\rho_L(i) = \rho_L^{\text{av}}$), the eigenmodes would agree on the long time scales, as revealed in Fig. 1(b), for the same model defined on a torus, with bulk dissipation. In this case indeed, ρ_L is spatially uniform and the Lyapunov spectrum corresponds to normal diffusion on the long time scales.

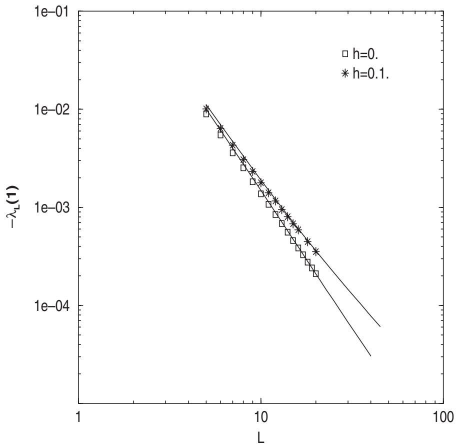

Fig. 2. First Lyapunov exponent $\lambda_L(1)$ versus L for $E_c = 2.2$, $\epsilon = 0.1$, $h = 0$, and $h = 0.1$. The theoretical values given by are drawn by solid lines.

For the shortest time scales (time scales of the order of one step of the dynamics), the diffusion is *singular*. This is due to the presence of thresholds. The corresponding modes correspond to directions of fast re-stabilization on the attractor.[19] On long time scales the effect of the threshold is averaged and smoothed (it acts in ρ_L), but on small time scales (of the order of the microscopic time) it induces singularities in the dynamics.

1.3.3. Stationarity Equations

It is possible to obtain an approximate analytic expression for ρ_L by noting that

$$\bar{\mathbf{F}}_L - \bar{\mathbf{X}}_L = \int \{(\gamma - 1)\,\epsilon \mathbf{Z}(\mathbf{X}) * \mathbf{X} + \alpha \Delta [\mathbf{Z}(\mathbf{X}) * \mathbf{X}]\}\, d\hat{\mu}_L(\hat{\mathbf{X}}) \tag{16}$$

is the average energy lost by each site in each time step of the dynamics in the stationary regime. At stationarity, this loss is compensated by the average entering energy flux $\bar{\Phi}_L$. The balance equation has the form

$$(\gamma - 1)\,\epsilon \bar{\mathbf{U}}_L + \alpha \Delta \bar{\mathbf{U}}_L = -\bar{\Phi}_L, \tag{17}$$

where $\bar{\mathbf{U}}_L \overset{\text{def}}{=} \rho_L * \bar{\mathbf{X}}_L^+$. The vector $\bar{\mathbf{X}}_L^+ = \{\bar{X}_L^+(i)\}_{i=1}^N$ is such that the ith component, $\bar{X}_L^+(i) \overset{\text{def}}{=} E[X_i | X_i \geqslant E_c]$, is the *conditional expectation* of X_i given that $X_i \geqslant E_c$. (Alternatively, $\bar{X}_L^+(i)$ is the average energy of X_i given that X_i is active.)

It is easy to solve (16) by decomposing $\bar{\mathbf{U}}_L$ on the eigenmodes of the Laplacian. ρ_L can then be obtained by noting that the spatial fluctuations of $\mathbf{X}_L^+$ are small (except near the boundary). One therefore considers $\mathbf{X}_L^+$ as spatially constant. This yields the following approximation for ρ_L:

$$\rho_L(\mathbf{x}) = \sum_{\mathbf{n}} A_{\mathbf{n}} \prod_{i=1}^{d} \sin(k_i x_i), \tag{18}$$

where $\mathbf{x} = (x_1,..., x_n)$ is the set of coordinates of a point in a d-dimensional lattice, $\mathbf{n} = (n_1,..., n_d)$ is the set of quantum numbers parameterizing the eigenmodes of the discrete Laplace operator, and

$$s_{\mathbf{n}} = \left[(\gamma - 1)\,\epsilon + 2\alpha \left(\sum_{i=1}^{d} \cos(k_i) - d \right) \right] \tag{19}$$

is the corresponding eigenvalue with $k_i = \frac{n_i \pi}{L+1}$.

The coefficients $A_{\mathbf{n}}$ are given by

$$A_{\mathbf{n}} = -\frac{2^d \bar{\omega}_L}{E^+_{\text{tot}}(L+1)^d} \frac{\prod_{i=1}^d C_{n_i}}{s_{\mathbf{n}}},$$

where $E^+_{\text{tot}} = \sum_{i=1}^N \bar{X}^+_L(i)$ and,

$$C_{n_i} = \sum_{x=1}^{L} \sin(k_i.x) = (-1)^{m_i} \frac{\sin(\frac{n_i \pi L}{2(L+1)})}{\sin(\frac{n_i \pi}{2(L+1)})}, \tag{20}$$

where $n_i = 2m_i + 1$.

The density of active sites $\rho_L^{\text{av}} = \frac{1}{N}\sum_{i=1}^N \rho_L(i)$ is given by

$$\rho_L^{\text{av}} = -\frac{\bar{\omega}_L}{L^d E^+_{\text{tot}}} \gamma_L, \tag{21}$$

where

$$\gamma_L = \left(\frac{2}{L+1}\right)^d \sum_{\mathbf{n}} \frac{\prod_{i=1}^d C^2_{n_i}}{s_{\mathbf{n}}}. \tag{22}$$

Equations (18) and (21) are in very good agreement with the empirical values. In Fig. 3(a) we plot the empirical and theoretical values of ρ_L for $L = 40$, $E_c = 2.2$, $\epsilon = 0.1$, and $h = 0.1$ and in Fig. 3(b) the empirical and theoretical curve ρ_L^{av} as a function of L, for $E_c = 2.2$, $\epsilon = 0.1$, $h = 0$ and $h = 0.1$.

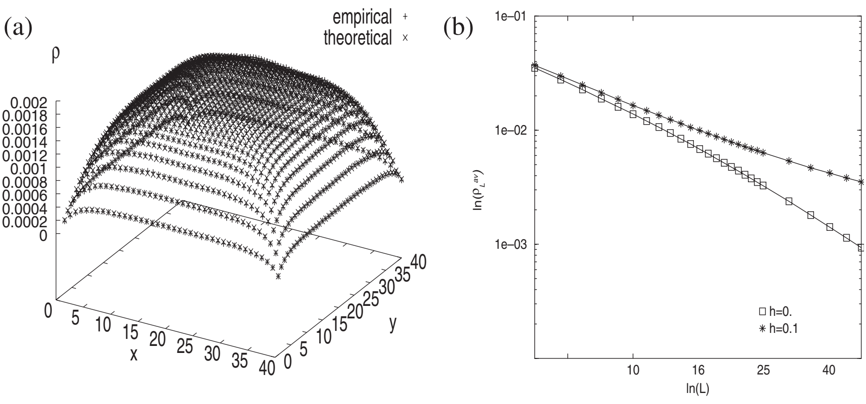

Fig. 3. (a) Empirical and theoretical values of ρ_L for $L = 40$, $E_c = 2.2$, $\epsilon = 0.1$, $h = 0.1$. (b) The empirical and theoretical (solid lines) curve representing ρ_L^{av} as a function of L, for $E_c = 2.2$, $\epsilon = 0.1$, $h = 0$, and $h = 0.1$.

Equation (21) is a balance equation. It can indeed be rewritten as

$$\rho_L^{\mathrm{av}} \bar{e}_L = \bar{\phi}_L, \tag{23}$$

where we recall that $\bar{\phi}_L = \frac{\varpi_L}{N}$ is the average energy injected per unit time and per site (see Eq. (9)) and

$$\bar{e}_L = \frac{E_{\mathrm{tot}}^{+}}{|\gamma_L|} \tag{24}$$

corresponds (from Eq. (23)) to the average energy dissipated per unit time and per active site. In the literature it is called the *dissipation rate*.[29]

Equation (23) implies that the susceptibility (12) obeys:

$$\chi_L = \frac{1}{\bar{e}_L} - \frac{\bar{\phi}_L}{\bar{e}_L^2} \frac{\partial \bar{e}_L}{\partial \bar{\phi}_L} = \frac{1}{\bar{e}_L} \left[1 - \rho_L^{\mathrm{av}} \frac{\partial \bar{e}_L}{\partial \bar{\phi}_L} \right]. \tag{25}$$

Were $\bar{\phi}_L$ and the dissipation rate $\bar{e}_L$ to be independent, this relation would reduce to $\chi_L = \frac{1}{\bar{e}_L}$. This corresponds to the result found by Vespignani and Zapperi in a sandpile model where the excitation rate per site and dissipation rate were independent tunable parameters.[29] Relation (25) is an extension of this result to the case where $\bar{e}_L$ and $\bar{\phi}_L$ are not independent and tunable parameters but are determined by the microscopic evolution. (This is exactly what is meant by "self-organized.")

As $L \to \infty$ it is easy to see that the leading contribution in (22) corresponds to modes $\mathbf{n} = (n_1,..., n_d)$ such that $n_k < \beta(L+1)$ for some $\beta > 0$ which can be taken arbitrary small. Indeed, for these modes, according to Eqs. (20) and (26), $C_{n_i} \sim \frac{2(L+1)}{n_i \pi}$ while $s_{\mathbf{n}} \sim (\gamma - 1)\,\epsilon - \frac{\pi^2}{(L+1)^2} \sum_{i=1}^{d} n_i^2$. Consequently, the sum on these modes scales like $(L+1)^{2d} \sum^{*} \frac{(2/\pi)^{2d}}{n_i^{2d}[(\gamma-1)\,\epsilon - \alpha((n_i \pi)/(L+1))^2]}$ where $\sum^{*}$ denotes the sum on the $\mathbf{n}$'s such that $\sup_k n_k < \beta(L+1)$. On the other hand, one can decompose the remaining terms in the sum defining γ_L into sums on the modes $\mathbf{n}$ so that exactly r components are smaller than $\beta(L+1)$. Each sum scales like $(L+1)^{2r}$, where $r < d$. Therefore, as $L \to \infty$, $|\gamma_L|$ scales like $a\,\frac{(L+1)^d}{2(1-\gamma)\,\epsilon}$ when $h > 0$, and like $c(L+1)^{d-2}$ when $h = 0$, with a and c nonnegative constants. Furthermore, $E_{\mathrm{tot}}^{+} \sim L^d$ since $E_c \leqslant \bar{X}_L^{+}(i) < K$, $i = 1 \cdots N$, where K is some constant independent of L. Therefore, when $L \to \infty$ the dissipation rate, $\bar{e}_L$, obeys the scaling relation

$$\bar{e}_L \sim \bar{X}_L^{+}[2(1-\gamma)\,\epsilon A + C L^{-2}], \tag{26}$$

where A and C are positive constants. Consequently, when $h > 0$, the dissipation rate *converges to a positive value as* $L \to \infty$, while it converges to 0

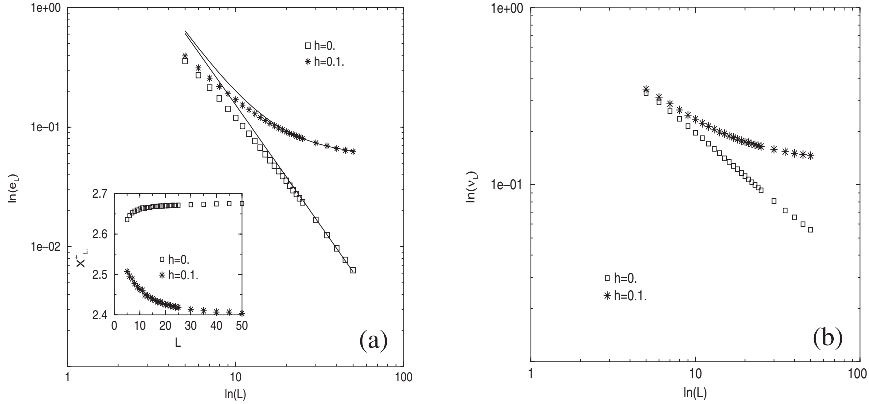

Fig. 4. (a) Dissipation rate versus L for $E_c = 2.2$, $\epsilon = 0.1$ for $h = 0$ and $h = 0.1$ (inset: corresponding values of $\bar{X}^+_L$). The solid lines represent the fitting curves $\bar{X}^+_L CL^{-2}$ (conservative case $h = 0$) and $\bar{X}^+_L[a + CL^{-2}]$ (nonconservative case $h = 0.1$) corresponding to Eq. (26). (b) shows the corresponding graph for $\bar{\omega}_L$.

like L^{-2} when $h = 0$. In Fig. 4(a) we plotted the dissipation rate $\bar{e}_L$. The fitting curves $\bar{X}^+_L CL^{-2}$, conservative case ($h = 0$), and $\bar{X}^+_L[A + CL^{-2}]$, nonconservative case ($h = 0.1$), corresponding to Eq. (26), are drawn in solid lines. A and C were obtained by regression. In Fig. 4(b) the excitation rate $\bar{\omega}_L$ is represented.

Note that there exists a stationary regime only if the dissipation rate is positive. Henceforth, there exists an h_L given by $\sum_{\mathbf{n}} \frac{\prod_{i=1}^d C^2_{n_i}}{s_{\mathbf{n}}} = 0$ such that for $h < h_L$ there is no stationary regime. h_L behaves asymptotically like $\frac{\log(1-(CL^{-2}/2\epsilon))}{\log(\epsilon)}$ and converges to 0 as $L \to \infty$.

1.3.4. Distributions of Avalanche Observables

Since the energy used to initiate an avalanche is dropped locally, $\xi^n_L(h)$, the maximal value that the random variable n can take, among all the avalanches having a nonzero probability of occurrence in the stationary regime, in a lattice of size L, is the largest scale where the effect of a local perturbation is observed, within one avalanche. Thus it corresponds to a correlation length *within one avalanche* and is a function of h. As discussed above, $\xi^n_L(h) < \infty$ for $h \geqslant 0$, $L < \infty$.

In the subcritical regime $h > 0$ the avalanche size (resp. duration, area, etc.) is bounded $\forall L$. Indeed, consider the evolution, under the singular diffusion (1), of a configuration $\mathbf{Y}$ in the *infinite* lattice[10] $\mathbb{Z}^d$, such that

[10] With bulk dissipation since $h > 0$.

$E_c > Y_i \geqslant E_c - \eta$, $\eta > 0$, $\forall i \in \mathbb{Z}^d$, with excitation at some point $i_0 \in \mathbb{Z}^d$. Consider simultaneously the normal damped diffusion $\mathbf{Y}'(t+1) = \mathbf{Y}'(t) + (\gamma - 1)\,\epsilon \mathbf{Y}'(t) + \alpha \Delta[\mathbf{Y}'(t)]$ on $\mathbb{Z}^d$ with a source $\delta = 1$ applied at time $t = 0$ at the site i_0, where $Y'_i(t) = \max(Y_i(t) - (E_c - \eta), 0)$. By definition $Y_i(t) \leqslant Y'_i(t) + E_c - \eta$, $\forall t \geqslant 0$. $\mathbf{Y}'(t)$ converges asymptotically to 0 and it is easy to show[11] that there exists both a bounded region $\mathcal{R} \subset \mathbb{Z}^d$ containing i_0 and a time t_0 such that, $\forall t \geqslant t_0$ and $\forall i \in \mathbb{Z}^d \backslash \mathcal{R}$, $0 \leqslant Y'_i(t) < \frac{\eta}{2}$. Since $Y_i(t) \leqslant Y'_i(t) + E_c - \eta$, $Y_i(t) \leqslant E_c - \frac{\eta}{2} < E_c$, $\forall t \geqslant t_0$, $\forall i \in \mathbb{Z}^d \backslash \mathcal{R}$. Consequently, the avalanche $\mathcal{C}(\mathbf{Y})$ does not go beyond $\mathcal{R}$ and the duration (resp. area, size) of $\mathcal{C}(\mathbf{Y})$ is bounded. Now, since the largest avalanche size in a finite box $\Lambda \subset \mathbb{Z}^d$ is generated from a stable configuration $\mathbf{X}$ such that $X_i < E_c$ it is possible to find a configuration $\mathbf{Y}$ as above such that $\mathcal{C}(\mathbf{X}) \subset \mathcal{C}(\mathbf{Y})$, provided η is sufficiently small ($\eta < E_c - \max_{i \in \Lambda} X_i$). Consequently, $\xi^n_L(h) < n(\mathcal{C}(\mathbf{Y})) < \infty$, $\forall L$, $\forall h > 0$. It follows that the maximal size, area, etc. are bounded, together with the corresponding average values, when $h > 0$. Moreover, the energy excitation rate, $\bar{\omega}_L(h)$, scales like $\frac{1}{\langle \tau \rangle_{L,h}}$. Consequently, $\bar{\omega}_L(h)$ converges, as $L \to \infty$, to a *strictly positive* value in the subcritical regime (see Fig. 4(b)).

On the other hand, the average avalanche observable and the corresponding correlation lengths *diverge* in the conservative case when $L \to \infty$. First, the behavior of the average avalanche size $\langle s \rangle_{L,h}$, as a function of h and L is obtained from a stationarity condition ensuring the balance between the incoming energy flux ($\delta = 1$ is injected each time an avalanche ends) and the average energy flux dissipated within an avalanche ($\langle s \rangle_{L,h} \bar{e}_L(h)$). This gives

$$\langle s \rangle_{L,h} \sim \frac{1}{\bar{e}_L(h)} \sim \frac{1}{\bar{X}^+_L[2(1-\gamma)\,\epsilon A + CL^{-2}]}. \tag{27}$$

Therefore $\langle s \rangle_{L,h}$ converges to a constant when $h > 0$, while it *diverges* like L^2 in the conservative case.[19] The adiabaticity condition imposes that the number of active sites at the tth avalanche step be bounded from above by a constant independent of L. This implies that the expectation of the avalanche observables a and τ also diverges. It is numerically observed that $\langle n \rangle_L$ diverges like

$$\langle n \rangle_L \sim L^{\gamma_n}, \tag{28}$$

where $n = a, s, t$, and $\gamma_n > 0$ in the conservative case.

[11] On $\mathbb{Z}^d$ the eigenmodes of the damped diffusion equation are $\lambda(k) = (\gamma - 1)\,\epsilon - \alpha k^2$, where k is the wave vector in the Fourier space. The presence of the damping coefficient $(\gamma - 1)\,\epsilon$ ensures the existence of the region $\mathcal{R}$.

Since most of the results available (in particular the numerical ones) are obtained for finite L, one has to propose a finite-size scaling form allowing an extrapolation to $L \to \infty$ from finite-size lattices results. The most common scaling Ansatz (see, in particular ref. 29) is the following (adapted to our notations):

$$P_L(n, h) = n^{-\tau_n} f\left(\frac{n}{\xi_L^n(h)}\right), \qquad n > n_0, \tag{29}$$

where $n_0 > 0$ is model-dependent.

This scaling form is discussed in more detail in Section 2.5. This is the analog of the finite-size scaling used in critical phenomena. Note that Eq. (29) implies $\langle n \rangle_{L,h} \sim \xi_L^n(h)^{(2-\tau_n)}$. Therefore, in the conservative case, $\xi_L^n(0) \sim L^{\beta_n}$, where

$$\beta_n = \frac{\gamma_n}{2-\tau_n}$$

and $P_L(n) = n^{-\tau_n} f(\frac{n}{L^{\beta_n}})$. This corresponds to the finite-size scaling Ansatz proposed by Kadanoff *et al.*[13] for SOC systems. Clearly, in this scaling form, τ_n and β_n are characteristic exponents allowing a determination of the universality class of $P^*(n)$.

Equation (29) also implies that the correlation length of *avalanche size* is a function of the dissipation rate with a scaling:

$$\xi_L^s(h) = \bar{e}_L(h)^{-\frac{1}{\sigma}}, \tag{30}$$

where $\sigma = 2 - \tau_s$. Therefore, for $h = 0$, the correlation length diverges like L^{β_s}, where $\beta_s = \frac{2}{2-\tau_s}$, and, for $h > 0$, $\xi_L^s(h)$ converges to a constant $\xi^s(h) = (2A(1-\gamma)\,\epsilon)^{-\frac{1}{\sigma}}$. Finally, as $h \to 0$, $\xi^s(h) \sim h^{-\delta}$, where $\delta = \frac{1}{\sigma}$. Consequently, δ is a characteristic exponent related to the singularity of correlation length $\xi_L^s(h)$ as $h \to 0$. In Section 2.7 we show how the exponents τ_n, β_n, and σ can be determined from the behavior of the zeros of a suitable partition function in full analogy with the usual critical phenomena.

From Eqs. (21) and (26) the density of the active sites behaves asymptotically like:

$$\rho_L^{\text{av}} \sim \frac{\bar{\omega}_L}{E_{\text{tot}}^+[2(1-\gamma)\,\epsilon A + C L^{-2}]}. \tag{31}$$

Consequently, the density of the active site converges to 0 for any $h \geqslant 0$ and scales like $\frac{1}{E_{\text{tot}}^+} \sim L^{-d}$ for $h > 0$ and like $\frac{\bar{\omega}_L}{L^{d-2}} \sim L^{2-d-\gamma_\tau}$ for $h = 0$, where $\langle \tau \rangle_L \sim L^{\gamma_\tau}$.

Finally, let us discuss the behavior of the susceptibility (25). It behaves like $\frac{1}{\bar{e}_L} = \langle s \rangle_L$ as $L \to \infty$ (unless the term $\rho_L^{\mathrm{av}} \frac{\partial \bar{e}_L}{\partial \bar{\phi}_L}$ converges to 1 as $L \to \infty$). This extends the relation already found by Vespignani and Zapperi[29] for sandpiles to the Zhang model.

1.3.5. Extrapolation to $L \to \infty$

Let us now make several important remarks resulting from the analysis of the previous sections. For $h = 0$, when $L \to \infty$ the system reaches a state where the correlation lengths ξ_L^n, $n = a, \tau, s$ diverge, and where the avalanches are statistically distributed according to a power law. In this sense, this corresponds to a critical state, reached spontaneously by the sole effect of the relaxation dynamics, of the adiabaticity condition in the energy excitation, and of the vanishing of the boundary dissipation rate. From the dynamical point of view, the positive Lyapunov exponent $\bar{\omega}_L \log(N)$, which is also the Kolmogorov–Sinai entropy, vanishes since the excitation rate $\bar{\omega}_L$ tends to zero. Therefore the asymptotic state has zero entropy. Correspondingly, the first negative Lyapunov exponent tends to zero (see Eq. (15)). Consequently, the Zhang model *loses its hyperbolic structure in the limit* $L \to \infty$. This is clearly expected since the loss of hyperbolicity is a necessary condition for algebraic correlation decay.

For $h > 0$, the correlation lengths remain bounded and the positive Lyapunov exponent (the entropy) diverges like $\log(N)$ since $\bar{\omega}_L(h)$ tends to a strictly positive value for $h > 0$. Hence, h can be used as a control parameter allowing a tuning of the system to the critical regime. In Section 2.7 we show that the corresponding phase transition can be handled by analyzing the Lee–Yang zeros of the proper partition function. The next part is devoted to the construction of the thermodynamic formalism for the Zhang model.

2. THERMODYNAMIC FORMALISM

In this part, we construct a symbolic encoding of the dynamics by a finite Markov chain, in a way such that each symbol corresponds to energy configurations undergoing *the same avalanche when a specified site is excited.* Hence, the coding is relevant not only for a characterization of the microscopic evolution but also for a description of the avalanche dynamics. The first SOC conjecture implies that there is a unique invariant measure in the code space for the associated Markov chain corresponding to the SRB measure (Section 2.5). By definition, it characterizes the microscopic dynamics. In particular, its support corresponds to the so-called "SOC attractor."[1, 3] But, by construction, it also characterizes the macroscopic avalanche distribution. This measure is a particular example of Gibbs measures in the sense of Sinai–Ruelle–Bowen. We construct other examples of these Gibbs measure in Section 2.5, by changing the weights in the Markov transition matrix, and discuss their connections with the microscopic dynamics and with the distribution of avalanche observables.

2.1. The Return Maps

We first redefine the Zhang model dynamics in terms of return maps. The avalanche, after the excitation of a site, maps unstable to stable configurations. One can view this process as a mapping from $\mathcal{M} \to \mathcal{M}$ where one includes the excitation process. Let T_i be the map $\mathcal{M} \to \mathcal{M}$ which associates to a stable energy configuration $\mathbf{X}$ the next stable configuration resulting from an avalanche obtained by the excitation of the site i in the configuration $\mathbf{X}$. Then:

$$T_i(\mathbf{X}) = F^{\tau(i,\mathbf{X})}(\mathbf{X} + \delta \mathbf{e}_i), \qquad \mathbf{X} \in \mathcal{M}, \tag{32}$$

in terms of the mapping of Eq. (1). The quantity

$$\tau(i, \mathbf{X}) \stackrel{\text{def}}{=} \inf\{t \geqslant 0, F^t(\mathbf{X} + \delta \mathbf{e}_i) \in \mathcal{M}\}$$

is the duration of the avalanche obtained by exciting the site i in the stable configuration $\mathbf{X}$. Note that $T_i(\mathbf{X})$ is a simple translation in the case where the excitation of i does not render it active.

From the mappings T_i, $i \in \Lambda$, we construct a new dynamical system where the phase space is $\Omega = \Sigma_\Lambda^+ \times \mathcal{M}$, where, as above, Σ_Λ^+ is the set of right infinite excitation sequences $\tilde{a} = \{a_1, \ldots, a_k, \ldots \mid a_k \in \Lambda\}$ but, this time, $\mathbf{X}$ is a *stable* energy configuration and $\hat{\mathbf{X}} = (\tilde{a}, \mathbf{X})$ is the corresponding point in Ω. Let π_1 and π_2 be the canonical projections respectively on Σ_Λ^+ and M (namely $\pi_1(\hat{\mathbf{X}}) = \tilde{a}$, $\pi_2(\hat{\mathbf{X}}) = \mathbf{X}$).

The evolution is now determined by a dynamical system of *skew product type*, $\mathscr{T}: \Omega \longrightarrow \Omega$ such that:

$$\mathscr{T}(\hat{\mathbf{X}}) \stackrel{\text{def}}{=} (\sigma\tilde{a}, T_{a_1}(\mathbf{X})); \qquad \hat{\mathbf{X}} \stackrel{\text{def}}{=} (a, \mathbf{X}). \tag{33}$$

Set $\hat{\mathbf{X}}(t) = \mathscr{T}^t(\hat{\mathbf{X}})$ and $\mathbf{X}(t) = \pi_2(\mathscr{T}^t(\hat{\mathbf{X}}))$. $\mathbf{X}(t)$ is now the stable energy configuration obtained after t avalanches, starting from the energy configuration $\mathbf{X}$, when the excitation sequence is $\tilde{a} = \pi_1(\hat{\mathbf{X}})$.

Due to the particular structure of the mapping (1), each map T_i, $i \in \Lambda$, is a *piecewise-affine map*.[18] Denote by $T_{(i,j)}$ the jth affine component[12] of

[12] The maps $T_{(i,j)}$ play, in some sense, the role of the toppling operators introduced by Dhar for Abelian sandpiles.[25] However, the structure of the Zhang model is more complex since these operators neither commute nor preserve the Lebesgue measure.

the map T_i. Denote $M_{(i,j)}$ the domain of definition of $T_{(i,j)}$. There is a one-to-one correspondence between the set of avalanches and the set of affine mappings. $T_{(i,j)}$ maps the energy configurations in $M_{(i,j)}$ to the stable configurations resulting from the avalanche (i, j).

The set of points $\mathscr{S}$ where $\mathscr{T}$ is not continuous is called the *singularity set*. This is a union of hyperplanes[13] constituting the borders $\partial\mathscr{M}_{(i,j)}$ of the $M_{(i,j)}$'s. The singularity set $\mathscr{S}$ is therefore the set of stable energy configurations such that some sites have an energy *exactly equal to* E_c at some time during some avalanche. These configurations have, therefore, some site "right at the threshold" at some time of their evolution. They are, therefore, highly sensitive to an infinitesimal perturbation on the energy of these sites. $\mathscr{S}$ plays an important role in the dynamics, which is discussed in the next section.

It can be proved that each map $T_{(i,j)}$ is a quasi-contraction. More precisely, it contracts on the subspace generated by the canonical basis vectors corresponding to the active sites and is neutral on the remaining part of $\mathbb{R}^N$. Furthermore,

$$\det(T_{(i,j)}) = \epsilon^{s((i,j))}, \tag{34}$$

where $s((i,j))$ is the size of the corresponding avalanche.[18]

Since Ω has a product structure, and since the excitation is independent of the dynamics on energy configurations, the invariant measures which we consider decompose as $\hat{\mu}^u_L \times \hat{\mu}^s_L$, where $\hat{\mu}^u_L$ is the induced measure on the unstable direction or *excitation* measure, and $\hat{\mu}^s_L$ is the induced measure on $\mathscr{M}$, or measure on the stable energy configurations. In the Zhang model $\hat{\mu}^u_L$ is the uniform Bernoulli measure since the successive excited sites are chosen independently with a fixed rate $\frac{1}{N}$. To the SRB measure defined in Section 1.3.2 corresponds an SRB measure for Eq. (33) with product structure $\hat{\mu}^u_L \times \hat{\mu}^s_L$. In the following we will keep the notation $\hat{\mu}_L$ for the SRB measure of the dynamical system (33).

2.2. Local Stable Manifolds

The construction of a Markov partition requires that almost every point have a local stable and unstable manifold.[14] Recall that the local stable manifold $\mathscr{W}^s_{\text{loc}}(\hat{\mathbf{X}})$ of $\hat{\mathbf{X}} \in \Omega$ is the largest connected set such that $\forall \hat{\mathbf{Y}} \in \mathscr{W}^s_{\text{loc}}$, $d(\mathscr{T}^t(\hat{\mathbf{Y}}), \mathscr{T}^t(\hat{\mathbf{X}})) \to 0$ when $t \to \infty$, d being a distance on Ω. Therefore, the trajectory of any point belonging to $\mathscr{W}^s_{\text{loc}}(\hat{\mathbf{X}})$ is asymptotically identical to

[13] The number of singularity planes is finite when $L < \infty$, but it tends to infinity as $L \to \infty$ when $h = 0$.

[14] By construction, each point has a local unstable manifold, the set Σ^+_A.

the trajectory of $\hat{\mathbf{X}}$, and these points are equivalent from the dynamical point of view. Equivalently, if $\hat{\mathbf{X}}$ has a local stable manifold of diameter larger than some $\eta > 0$ then a small perturbation of size $\leqslant \eta$ in the configuration space will be asymptotically damped.

Were the map $\mathscr{T}$ to be regular, the existence of local stable manifolds would be ensured by the standard theorems on regular ($\mathscr{C}^{1+\alpha}$) hyperbolic dynamical systems.[31] However, $\mathscr{T}$ is not continuous on the singularity set $\mathscr{S}$ and some points may have no local stable manifolds.

The main problem is to estimate the $\hat{\mu}_L$ measure of the "bad" points having no local stable manifold. The following result is useful.

Proposition 1. Let $\mathscr{U}_\delta(\mathscr{S}) = \{\hat{\mathbf{Y}} \in \Omega \mid d(\hat{\mathbf{Y}}, \mathscr{S}) \leqslant \delta\}$ be the η-neighborhood of $\mathscr{S}$. Assume that for $\eta > 0$ sufficiently small:

$$\hat{\mu}_L[\mathscr{U}_\eta(\mathscr{S})] \leqslant C\eta^\alpha \tag{35}$$

for $C > 0$ and $\alpha > 0$. Then $\hat{\mu}_L$ almost every point has a local stable manifold of positive diameter.

Proof. The existence of a local stable manifold of positive diameter for a point $\hat{\mathbf{X}}$ in the support of $\hat{\mu}_L$ is ensured if one can find a number $0 > \gamma > \lambda_L(1)$ and a time $t_0 < \infty$ such that $\forall t > t_0$, $d(\mathscr{T}^t \hat{\mathbf{X}}, \mathscr{S}) > e^{\gamma t}$. Indeed, in this case a sufficiently small ball around $\hat{\mathbf{X}}$ contracts asymptotically faster than it approaches the singularity set, and, consequently, all the points of this ball belong to the stable manifold of $\hat{\mathbf{X}}$. Consequently, the set of bad points is included in the set:

$$\{\hat{\mathbf{X}} \mid \forall t_0 \geqslant 0, \exists t \geqslant t_0, d(\mathscr{T}^t \hat{\mathbf{X}}, \mathscr{S}) \leqslant e^{\gamma t}\} = \bigcap_{t_0=0}^{\infty} \bigcup_{t=t_0}^{\infty} \mathscr{T}^{-t}(\mathscr{U}_{e^{\gamma t}}(\mathscr{S}))$$

for some γ such that $0 > \gamma > \lambda_L(1)$. From the Borel–Cantelli lemma the measure of this set is zero provided the series $\sum_{t=0}^{\infty} \hat{\mu}_L[\mathscr{U}_{e^{\gamma t}}(\mathscr{S})]$ converges. This is true if the condition (35) holds with $\eta = e^\gamma$. ▮

We have not yet established a mathematical proof of the condition (35) but we have strong numerical evidence. We have numerically computed (Fig. 5) $\mu_L\{\mathscr{U}_\eta(\mathscr{S})\}$ as a function of η for $L = 30$, $E_c = 2.2$, $\epsilon = 0.1$, $h = 0$, $h = 0.1$. We observe that $\mu_L\{\mathscr{U}_\eta(\mathscr{S})\}$ decays like η^α, as $\eta \to 0$, where $\alpha = 0.98 \pm 0.01$ for $h = 0$ and $h = 0.1$. This observation is supported by another well-known result, already in the early paper of Zhang. The energy density per site has the characteristic structure[15] depicted Fig. 5(b). More

[15] The remarkable peak structure of this distribution has interested many people, but no one has been able, up until now, to give an analytic form for it.

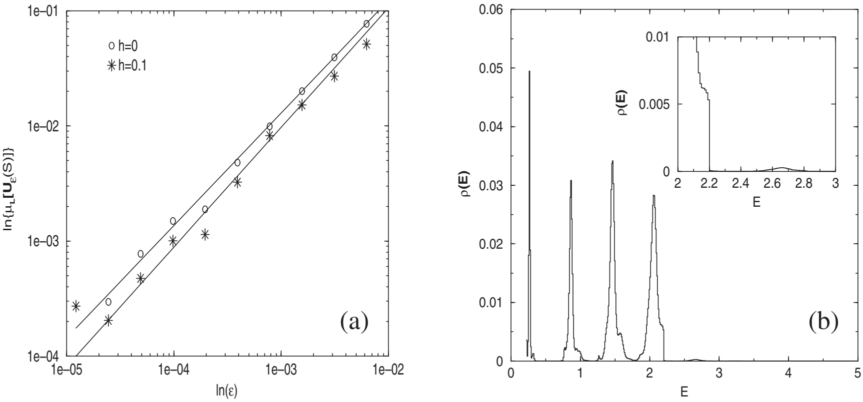

Fig. 5. (a) Probability $\mu_L\{\mathscr{U}_\eta(\mathscr{S})\}$ as a function of η for $L=30$, $E_c=2.2$, $\epsilon=0.1$, $h=0$, and $h=0.1$. (b) Probability $\rho(E)$, where $d=2$, $L=30$, $E_c=2.2$, $\epsilon=0.1$, and $h=0$. Inset: zoom around E_c.

precisely, in this figure, we have plotted $\rho(E) \stackrel{\text{def}}{=} \mu_L[\mathbf{X} \mid \exists i \in \Lambda, X_i \in [E, E+dE]]$. If we focus on a small neighborhood of E_c, we note that $\rho(E)$ is seemingly absolutely continuous on the left (explaining the exponent $\alpha \sim 1$).

The conclusions drawn from this simulation are quite fascinating. On one hand, they numerically support the assumption (35). On this basis, we will therefore assume in the following that $\hat{\mu}_L$ almost every point has a local stable manifold of diameter bounded from below. Then it is possible to generate a finite Markov partition $\mathscr{P}$ used for symbolic dynamics in the next section.

But on the other hand, this shows that the probability of approaching a small neighborhood of $\mathscr{S}$, *though weak, is nonzero*. Consequently, the Zhang model displays an interesting form of initial condition sensitivity. Indeed, if a trajectory approaches $\mathscr{S}$ sufficiently close at a time t, a small perturbation in the energy configuration at time t can induce a response *which is not proportional to the perturbation*. This happens if the perturbed trajectory crosses the singularity set since the avalanche will be different in the initial configuration and in the perturbed configuration. More precisely, if, in the trajectory of an energy configuration $\mathbf{X}$, a site i has its energy arbitrary close to E_c at some time t, then it is obvious that a small perturbation on this site can induce a completely different evolution. This phenomenon is particularly prominent since the measure of any η-neighborhood of the singularity set is positive for $\eta > 0$. This indeed means that, with positive probability, a trajectory will show wild sensitivity to arbitrarily weak perturbations for those times when it approaches the singularity set.

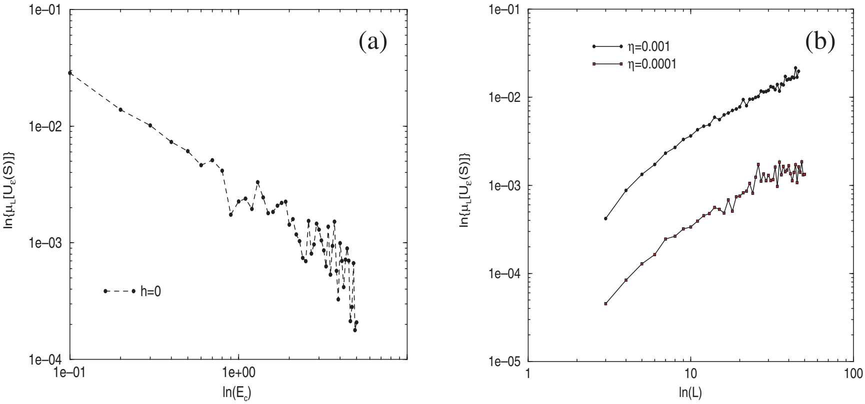

Fig. 6. Variation of $\mu_L\{\mathscr{U}_\eta(\mathscr{S})\}$ as a function of E_c for $L = 30$ (a) and as a function of L (b), for $\eta = 0.001$ and $\eta = 0.0001$, $E_c = 2.2$, $\epsilon = 0.1$, and $h = 0$. The statistics were done for 10 trajectories and 10^6 time steps per trajectory.

This results in an effective unpredictability of the evolution (weak initial condition sensitivity).

We have also numerically noticed (Fig. 6(b)) that $\mu_L\{\mathscr{U}_\eta(\mathscr{S})\}$ decreases with E_c and increases with L. This suggests that the initial condition sensitivity is more and more prominent as L grows. Note that the number of singularity planes diverges as $L \to \infty$ for $h = 0$.

Finally, the existence of local stable manifolds of sufficiently large diameter is a crucial step toward the proof of Conjecture 1. This allows us to form a Hopf chain (a path made of pieces of local stable and unstable manifolds between iterates of almost every point $\widehat{\mathbf{Z}}, \widehat{\mathbf{Y}}$ when they visit $\mathscr{U}_\eta(\widehat{\mathbf{X}})$). Moreover, it is shown in ref. 18, that, in many cases, there exists a topologically transitive orbit. By standard arguments from ergodic theory concerning the equality of forward and backward averages one can then prove that almost every pair of points on the invariant set belongs to the same ergodic component. This is exactly Conjecture 1. Consequently, the only missing step toward a proof of this conjecture is a mathematical validation of the numerical result presented in Fig. 5.

2.3. Markov Partition and Symbolic Dynamics

In this section, we construct a symbolic coding of the dynamics, relevant for a characterization of both the microscopic dynamics and the avalanches dynamics. One first has to construct a *finite* Markov partition of the phase space $\Omega = \Sigma_\Lambda^+ \times \mathscr{M}$.[28] The existence of a finite Markov partition is ensured by the existence of a local stable manifold of sufficiently large diameter for almost every point in Ω.[18] Moreover, in this case one may construct a Markov partition $\mathcal{P} = \{\mathcal{P}_\omega\}_{\omega \in \mathcal{I}}$, where $\mathcal{I}$ is finite, such that each domain $\mathcal{M}_{ij}$ is a finite union of the partition elements. By construction, to each element ω of this partition corresponds a sub-domain $\mathcal{P}_\omega \subset \mathcal{M}_{ij}$ and henceforth a unique avalanche (i, j). Equivalently, to each ω one can associate a double index $(i(\omega), j(\omega))$, where the first index refers to the site where the avalanche starts and the second to the corresponding avalanche. Note, however, that *several symbols* may correspond to the *same avalanche* since an avalanche (i, j) is usually represented by several partition-elements. However, in order to simplify the notation we will use ω and the avalanche $(i(\omega), j(\omega))$ whenever this causes no confusion.

To the Markov partition $\mathcal{P}$ we associate a transition matrix $\mathcal{A} = (\mathcal{A}_{ij})_{i, j \in \mathcal{I}}$ such that $\mathcal{A}_{ij} = 1$ if $\mathcal{P}_j \cap \mathcal{T}^{-1}(\mathcal{P}_i) \neq \emptyset$ (the transition $i \to j$ is legal) and $\mathcal{A}_{ij} = 0$ otherwise, and a transition graph $\mathcal{G}$ with vertices $\omega \in \mathcal{I}$ and oriented edges $i \to j$ for all pairs (i, j) such that $\mathcal{A}_{ij} = 1$. This provides a *symbolic dynamics* description of the Zhang model where the trajectory of a point is represented by a legal sequence of symbols $\cdots \omega_1 \omega_2 \cdots \omega_n \cdots$ corresponding to the partition elements that this point meets in its history. By construction we have the following properties:

(i) Every (legal) finite path $\omega_1,\dots, \omega_n$, $\omega_k \in \mathcal{I}$ corresponds to a *legal sequence of avalanches* $(i(\omega_1), j(\omega_1)),\dots, (i(\omega_n), j(\omega_n))$;

(ii) For any $\hat{\mathbf{X}}$, each orbit segment $\{\mathcal{T}^l(\hat{\mathbf{X}})\}_{1 \leqslant l \leqslant n}$ is realized as a path of length n on the Markov transition graph.

Denote by $\tilde{\omega} = \omega_{-n} \cdots \omega_{-1} \omega_0 \omega_1 \omega_2 \cdots \omega_n \cdots$, where $\mathcal{A}_{\omega_n, \omega_{n+1}} = 1$, $n \in \mathbb{Z}$, a legal bi-infinite sequence. Call $\mathcal{X}$ ($\mathcal{X}^+$) the set of bi-infinite (right infinite) legal sequences. There is a one-to-one correspondence, denoted by $\simeq$, between a symbolic sequence $\tilde{\omega}$ and a point $\hat{\mathbf{X}}$ in the phase space (up to a set of measure zero). Let $\pi = (\pi^+, \pi^-)$ be the corresponding conjugating mapping such that if $\tilde{\omega} \simeq \hat{\mathbf{X}} = (\tilde{a}, \mathbf{X})$, then $\pi^+(\tilde{\omega}) = \tilde{a}$ and $\pi^-(\tilde{\omega}) = \mathbf{X}$. Note that π^+ projects on the unstable direction (Σ_Λ^+), while π^- projects on the stable space $(\mathcal{M})$. Write $[\tilde{\omega}]_n$ for an n-cylinder (this is the subset of $\mathcal{X}$ where the sequences have the same n first digits as $\tilde{\omega}$). Denote by $\sigma_{\mathcal{A}}$ the shift on $\mathcal{X}$. The forward orbit of $\tilde{\omega}$ under $\sigma_{\mathcal{A}}$ encodes, by definition, the excitation sequence, and the backward orbit of $\tilde{\omega}$ encodes the point in the energy configuration space $\mathcal{M}$.

2.4. Markov Chain and the SOC State

We construct a Markov chain by defining a transition kernel $\mathcal{W}$ from the matrix $\mathcal{A}$. $\mathcal{A}$ has the following property. Let D_ω^+ be the set of follower elements of ω (and D_ω^- the set of predecessors of ω). $\# D_\omega^+ \geqslant N$, $\forall \omega$ since after the avalanche corresponding to ω one can excite any site in Λ. Moreover, by definition of the Markov partition, if $\beta, \gamma \in D_\omega^+$ and $\beta \neq \gamma$, then $i(\beta) \neq i(\gamma)$. Hence, $\#D_\omega = N, \forall \omega$. Consequently, $\mathscr{A}$ has *exactly N nonzero components per row*, corresponding to each of the N possible choices of an excited site, and the topological entropy is $\log(N)$.

Since the excited site of Λ is chosen with respect to the uniform Bernoulli measure the transition probability of the Markov partition for any edge from ω to D_ω^+ is constant, equal to $\frac{1}{N}$. Therefore the transition kernel is

$$\mathscr{W} = \frac{1}{N} \mathscr{A}. \tag{36}$$

As a consequence, $\mathscr{W}$ is a *sparse* matrix. Indeed, the number of symbols ω is at most equal to the number of possible avalanches, which is bounded from below by the largest avalanche size (duration, area) ξ_L^s. Since, for $h=0$, $\xi_L^s \sim L^{\beta_s}$, $\beta_s > d$, the proportion of nonzero entries on each row, $r_L = \frac{N}{\#\mathscr{I}} \simeq L^{d-\beta_s}$, tends to zero as $L \to \infty$. This remark justifies approximating the transition matrix by a *random* matrix. This could be used as an approximation to determine the spectral gap of $\mathscr{W}$ (see the discussion).

The set of symbols $\mathscr{I}$ decomposes into transient and recurrent nodes. Call $\mathscr{R}(\mathscr{I})$ the set of recurrent nodes and $\mathscr{N}_L \stackrel{\text{def}}{=} \#\mathscr{R}(\mathscr{I})$ the number of elements in $\mathscr{R}(\mathscr{I})$. The first SOC conjecture requires that the set of recurrent nodes be irreducible. This is *not* in contradiction with the sparseness of $\mathscr{W}$. For example, it can be proved that for sparse random matrices, with $K(m)$ uniformly distributed nonzero entries per row, where m is the size of the matrix, the set of nodes is almost-surely constituted by *one* irreducible recurrent cluster as m tends to infinity provided $K(m) > \log(m)$.[34]

If $\mathscr{W}$ is mixing,[16] Conjecture 1 writes, for the Markov chain,

$$m_L = \lim_{t \to \infty} \tilde{v} \mathscr{W}^t, \tag{37}$$

where v is any initial probability distribution on $\mathscr{I}$. Note that Eq. (37) is the analog of Eq. (7), footnote 4. In this case $\mathscr{W}$ has a unique eigenvalue 1 and a unique left eigenvector m_L, corresponding to the invariant probability distribution of the Markov chain.[35] $m_L(\omega)$ is nonzero only if $\omega \in \mathscr{R}(\mathscr{I})$. Furthermore, there is a spectral gap between the second eigenvalue with the

[16] There exists $n \geqslant 1$ such that, for each pair i, j, $\mathscr{W}^n(i, j) > 0$, i.e., there is a path connecting each node i, j.

largest modulus and 1. The gap gives the exponential correlation decay and also the rate of convergence to equilibrium. Since the Zhang model loses hyperbolicity as $L \to \infty$, it is expected that the gap vanishes for $h = 0$ and stays positive for $h > 0$ when $L \to \infty$.

The invariant probability distribution m_L characterizes the probability of occurrence of any recurrent symbol at stationarity. This is, therefore, the fundamental object characterizing the SOC state. In particular, the probability distribution of an avalanche observable is given by

$$P_L(n, h) = \sum_{\omega \in \mathcal{I}, n(\omega) = n} m_L(\omega). \tag{38}$$

Recall that $n(\omega)$ stands for $n(i(\omega), j(\omega))$. This is the value that the observable n takes in the avalanche $i(\omega), j(\omega)$. Note that m_L depends on h, but we dropped this dependence in order to simplify the notation. The moments of n are given by

$$\langle n^q \rangle_{L,h} = \sum_{\omega \in \mathcal{I}} n^q(\omega)\, m_L(\omega) = \sum_{n=0}^{\xi_L^n} n^q \sum_{\omega \in \mathcal{I}, n(\omega) = n} m_L(\omega) = \sum_{n=0}^{\xi_L^n} P_L(n, h)\, n^q. \tag{39}$$

More generally, the joint probability $\tilde{\mu}_L$ on the space of infinite symbolic sequences $\mathcal{X}$ is obtained from the Chapman–Kolmogorov equation. For any cylinder $[\tilde{\omega}]_T = \omega_1 \cdots \omega_T$:

$$\tilde{\mu}_L([\tilde{\omega}]_T) = m_L(\omega_1)\, \mathcal{W}_{\omega_1\omega_2} \mathcal{W}_{\omega_2\omega_3} \cdots \mathcal{W}_{\omega_{T-1}\omega_T}. \tag{40}$$

As is shown in the next section, $\tilde{\mu}_L$ is the measure of maximal entropy $\log(N)$. Also note that since $\hat{\mu}_L$, the SRB measure on Ω, has a constant Jacobian along unstable fibers (see next section), it is also the measure of maximal entropy. Therefore the measure $\tilde{\mu}_L$ projects down to $\hat{\mu}_L$ via $\hat{\mu}_L = \tilde{\mu}_L \circ \pi^{-1}$ with marginals $\hat{\mu}_L^u = \tilde{\mu}_L \circ \pi^+$ and $\hat{\mu}_L^s = \tilde{\mu}_L \circ \pi^-$. In the following we will therefore make no distinction between the average with respect to $\tilde{\mu}_L$ or with respect to $\hat{\mu}_L$. From the measure $\tilde{\mu}_L$ one can compute all the time correlations whence also the joint probability on the space of trajectories of $n(t)$ of the observable n (see Eq. (54)).

2.5. Thermodynamic Formalism, Gibbs Measures, and Generating Function of Avalanche Size Distribution

In this section, we use the thermodynamic formalism (see Appendix) to construct Gibbs measures by choosing different families of potentials.

The SRB measure $\hat{\mu}_L$ is the equilibrium state which maximizes the potential $-\log(\det(\pi^+(D\mathcal{T}_{\hat{X}})))$.[27] In the Zhang model this potential is a constant $-\log(N)$. It follows that the SRB measure maximizes the entropy. Since the maximal metric entropy is the topological entropy $\log(N)$, it also follows that the *SRB measure has zero pressure*. Moreover, according to Eq. (72), each cylinder $[\omega]_T$, of length T, has the same measure $\hat{\mu}_L[\omega]_T = \frac{1}{N^T}$. Therefore, $\hat{\mu}_L$ is *uniform on the attractor*. Then the probability of having an avalanche of size s (resp. duration τ, area a, etc.) is *proportional* to the number of symbols ω such that $s(\omega) = s$.

Note that the invariant measure is also uniform in the Abelian sandpile,(25) but in our case the attractor has a more complex structure. In the Abelian sandpile the attractor is a finite set of points in an N-dimensional torus,[17] while in the Zhang model this is a *fractal set*. More precisely, the dynamical system (33) is a probabilistic graph iterated function system (IFS) (see ref. 38 for a review) with maps[18] $\{T_\omega\}_{\omega \in \mathscr{I}}$, with transition graph $\mathscr{A}$ and where the probability of selecting a map $\{T_\omega\}_{\omega \in \mathscr{I}}$ is the SRB measure.(18) In the simplest cases, it is possible to construct the attractor of the Zhang model "by hand," by simple iteration of the IFS.(18)

It is well known that the thermodynamic formalism is a powerful tool to analyze the structure of fractal sets, by choosing suitable potentials.(38) It is remarkable that, in the Zhang model, these potentials also characterize properties of *avalanche distributions*. Let us introduce a first family of potentials:

$$\phi^1_{\beta,q}(\tilde{\omega}) = -q \log(N) + \beta \log(\det(DT_{\omega_1})), \tag{41}$$

where DT_{ω_1} stands for $DT_{(i(\omega_1), j(\omega_1))}$. For $q = 1, \beta = 0$ the corresponding Gibbs measure is the SRB measure. Tuning q and β allows one to select different Gibbs measures (singular with respect to the SRB), giving more information about the attractor structure. In particular, for *conformal* IFS, these potentials are used to compute the *multifractal* spectrum, which is the Legendre transform of the function $D(q) = \frac{\beta(q)}{q-1}$ (the q-Renyi dimension), where β is such that the corresponding topological pressure vanishes.(38)

$$\mathscr{F}^1_L(\beta(q), q) = 0. \tag{42}$$

Now, according to Eq. (34), $\log(\det(DT_{\tilde{\omega}_1})) = \log(\epsilon)\, s(\omega_1)$, namely the phase-space contraction is proportional to the *avalanche size*. Consequently:

$$\phi^1_{\beta,q}(\tilde{\omega}) = -q \log(N) + \beta \log(\epsilon)\, s(\omega_1). \tag{43}$$

[17] The toppling operators in the Abelian sandpile are translations in the phase space and the phase space is finite.

[18] Note that, although these maps are only *quasi*-contractions, any finite composition of these maps along the graph is a contraction.

The partition function is written as

$$Z_T^1(\beta, q) = \frac{1}{N^{Tq}} \sum_{\tilde{\omega} \in \chi_T^+} \prod_{t=1}^{T} \epsilon^{\beta s(\omega_t)}. \tag{44}$$

By introducing the transition matrix $\mathscr{A}$, one may rewrite $Z_T^1(\beta, q)$ as

$$Z_T^1(\beta, q) = \frac{1}{N^{Tq}} \sum_{s_1,\ldots, s_T} \epsilon^{\sum_{t=1}^{T} \beta s_t} \sum_{\tilde{\omega} \in \gamma_{s_1,\ldots, s_T}} \mathscr{A}_{\omega_1, \omega_2} \mathscr{A}_{\omega_2, \omega_3} \cdots \mathscr{A}_{\omega_{T-1}, \omega_T}$$

where $\gamma_{s_1,\ldots, s_T} = \{\tilde{\omega} \mid s(\omega_1) = s_1; s(\omega_2) = s_2;\ldots; s(\omega_T) = s_T\}$. Finally,

$$Z_T^1(\beta, q) = \frac{\mathscr{G}_T(\beta,\ldots, \beta)}{N^{T(q-1)}}, \tag{45}$$

where

$$\mathscr{G}_T(\beta_1,\ldots, \beta_T) = \sum_{s_1,\ldots, s_T} \epsilon^{\sum_{t=1}^{T} \beta_t s_t} \operatorname{Prob}[s_1,\ldots, s_T] \tag{46}$$

is the generating function of the T-time joint probability of avalanche sizes $\operatorname{Prob}[s_1,\ldots, s_T]$. As a consequence, the topological pressure $\mathscr{F}_L^1(\beta, q)$ may be used to compute the averages of the avalanche-size distribution.

The first important example follows from Eq. (73) (Appendix). The average phase-space contraction rate is given by

$$\left.\frac{\partial \mathscr{F}_L^1(\beta, q)}{\partial \beta}\right|_{\beta=0,\, q=1} = \log(\epsilon) \langle s \rangle_{L, h}. \tag{47}$$

This equation is analogous to the one giving the magnetization when deriving the free energy with respect to a uniform local field. This is an important result derived from the thermodynamic formalism since it relates the average volume contraction rate given by the sum of negative Lyapunov exponents[19] $\zeta_L(i)$ to the average avalanche size:

$$\sum_{i=1}^{N} \zeta_L(i) = \log(\epsilon) \langle s \rangle_{L, h}. \tag{48}$$

Now, since $\langle s \rangle_{L, h} \sim \bar{e}_L$, we have

$$\sum_{i=1}^{N} \lambda_L(i) \sim \log(\epsilon) \frac{\bar{\omega}_L}{\bar{e}_L} \tag{49}$$

[19] These are the Lyapunov exponents of the dynamical system (33) (return maps). Since $\frac{1}{\bar{\omega}_L}$ is the average return time $\zeta_L(i) = \frac{\lambda_L(i)}{\bar{\omega}_L}$, $i = 1 \cdots L^d$.

Therefore, in the Zhang model, *the average contraction rate in the phase space is given by the ratio between the excitation rate and the dissipation rate*. This is a prominent example of connection between a microscopic quantity and macroscopic observables.

The second derivative of the pressure with respect to β is

$$\left.\frac{\partial^2 \mathcal{F}^1_L(\beta, q)}{\partial \beta^2}\right|_{\beta=0,\, q=1} = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \langle s_{\omega_t} s_{\omega_0} \rangle_{L,h} - \langle s \rangle^2_{L,h}. \tag{50}$$

Hence, this equation gives the variance, along a typical trajectory, of the Gaussian fluctuations of the avalanche size (equivalently, of the contraction rate) around the average values. In Section 2.7 we discuss the failure of differentiability of $\mathcal{F}^1_L(\beta, q)$ for $h=0$ in the thermodynamic limit, which implies that $\frac{\partial^2 \mathcal{F}^1_L(\beta, q)}{\partial \beta^2}|_{\beta=0,\, q=1}$ diverges and that the central limit theorem is violated (as expected for a critical phenomenon).

More generally, Eq. (34) implies that the local volume contraction is distributed according to a truncated power law ($P_L(s)$). This certainly entails important properties for the structure of the attractor in the Zhang model. This is briefly discussed below and in more detail in ref. 39.

Equation (46) suggests that more detailed information about the probability distribution of avalanche sizes, or more generally of any other avalanche observable n, can be obtained by introducing in the potential the formal equivalent of a local magnetic field η, which allows, in statistical mechanics, the computation of the local magnetization and of the higher-order cumulants. A natural variation of (41) is the time-dependent potential:[20]

$$\phi_\eta(t, \tilde{\omega}) = -\log(N) + \eta_t n(\omega_t), \tag{51}$$

where $\eta = (\eta_1,\ldots, \eta_t,\ldots)$, and the partition function

$$Z_T(\eta_1,\ldots, \eta_T) = N^{-T} \sum_{\omega \in \chi_T^+} e^{\sum_{t=1}^{T} \eta_t n(\omega_t)}, \tag{52}$$

which gives (44) by setting $q=1$ and $\eta_t = \beta$. Set

$$\mathcal{F}_L(\eta) = \lim_{T\to\infty} \frac{1}{T} \log(Z_T(\eta_1,\ldots, \eta_T)); \tag{53}$$

[20] q is set to 1 here because this potential is used to compute averages with respect to the SRB measure. One may, however, consider a generalization where q is a tunable parameter.

then this function generates the k-time correlations of the observable n:

$$\left.\frac{\partial^k \mathcal{F}_L(\eta)}{\partial\eta_1 \cdots \partial\eta_k}\right|_{\eta=0} = \langle n_1; n_2; ...; n_k \rangle_{L,h}. \tag{54}$$

In particular, the cumulants $C_L(k)$ of the marginal distribution $P_L(n)$ are given by

$$\left.\frac{\partial^k \mathcal{F}_L(\eta)}{\partial\eta_1^k}\right|_{\eta_1=0} = C_L(k) = \left.\frac{\partial^k G_L(t)}{\partial t^k}\right|_{t=0}, \tag{55}$$

where

$$G_L(t) \stackrel{\text{def}}{=} \log(\langle e^{tn} \rangle_{L,h}) = \log\left(\sum_{n=0}^{\xi_L^n(h)} P_L(n,h)\, e^{tn}\right) \tag{56}$$

is the generating function of the cumulants of $P_L(n,h)$.

Now let us return to (41). It was mentioned in the beginning of this section that $\phi^1(\beta, q)$ is used to compute the multifractal spectrum of *conformal* IFS. As discussed above, this leads us to suspect that there might be some connection between the structure of the attractor and the avalanche-size distribution. This is mainly due to the fact that the local contraction rate is given by the avalanche size (Eq. (34)). However, the Zhang model is *not conformal* (the contraction is not uniform in the phase space, as is easily seen in the Lyapunov spectrum of Fig. 1). In spite of this, we believe that the potential (41) might be useful when considering the marginal energy distribution of one site. This point is under current investigation and will be published elsewhere (ref. 39).

It is, anyway, possible to compute the multifractal spectrum of non-conformal IFS with a (sub-additive[21]) thermodynamic formalism. Let $\alpha_i(\tilde{\omega}, k)$, $i = 1 \cdots N$, $k = 1 \cdots \infty$, be the singular values of the matrix $DT_{\omega_k} \cdots DT_{\omega_1}$ ordered such that $1 \geqslant \alpha_1(\tilde{\omega}, k) \geqslant \cdots \geqslant \alpha_N(\tilde{\omega}, k) > 0$ and note that there is a finite k such that, whatever $\tilde{\omega}$, $\forall l > k$, $1 > \alpha_1(\tilde{\omega}, l)$. Define:

$$g_\beta(\tilde{\omega}, k) = \alpha_1(\tilde{\omega}, k)\, \alpha_2(\tilde{\omega}, k)\, \alpha_{j-1}(\tilde{\omega}, k)(\alpha_j(\tilde{\omega}, k))^{\beta-j+1} \qquad \text{if} \quad 0 < \beta \leqslant N, \tag{57}$$

$$g_\beta(\tilde{\omega}, k) = (\det(T_{\omega_k} \circ \cdots \circ T_{\omega_1})^{\frac{\beta}{N}}) = \epsilon^{\frac{\beta}{N}\sum_{l=1}^{k} s(\omega_l)} \qquad \text{if} \quad \beta \geqslant N, \tag{58}$$

[21] The framework for such a generalized thermodynamic formalism is described in ref. 40 and in greater generality in ref. 41.

where, in Eq. (57), j is the integer such that $j-1<\beta\leqslant j$. Consider the (sub-additive) potential:

$$\phi^2_{\beta,q,k}(\tilde{\omega}) = -q\log(N)+\log(g_\beta(\tilde{\omega},k)) \tag{59}$$

and define the pressure by

$$\mathscr{F}^2_L(\beta,q) = \lim_{k\to\infty}\frac{1}{k}\log\sum_{\omega\in\chi_k^+}\phi^2_{\beta,q,k}(\tilde{\omega}). \tag{60}$$

Then there exists a unique $\beta\equiv\beta(q)$ such that the topological pressure $\mathscr{F}^2_L$ vanishes.[42] The numbers $D(q)=\frac{\beta(q)}{q-1}$ are the Renyi dimensions, and the multifractal spectrum is the Legendre transform of $D(q)$.

We are currently investigating the relations between this and macroscopic transport properties.[39] In particular, we are trying to interpret the singular values in terms of avalanche properties. In fact, the singular value $\alpha_i(\tilde{\omega},t)$ is the local contraction rate, at time t, along the trajectory $\tilde{\omega}$, in a direction corresponding to the ith eigenvector of the matrix $\tilde{DT}_{\omega_1}\cdots\tilde{DT}_{\omega_t}\cdot DT_{\omega_t}\cdots DT_{\omega_1}$, where $\tilde{\ }$ denotes the transpose. The matrix DT_{ω_1} gives exactly the redistribution of energy after the avalanche ω_1. Consequently, the $\alpha_i(\tilde{\omega},t)$'s are the singular values of the t time step "transport matrix" $DT_{\omega_t}\circ\cdots\circ DT_{\omega_1}$ when the trajectory in the phase space is $\tilde{\omega}$. In particular, it is well known that $\frac{1}{t}\log\alpha_i(\tilde{\omega},t)\to\zeta_L(i)$ almost-surely, where $\zeta_L(i)$ is the ith Lyapunov exponent of the dynamical system (33). But we have seen in Section 2.3.2 that the Lyapunov exponents are related to energy transport. Consequently, we believe that there is a strong connection between the energy transport properties in the lattice, on one hand, and the fractal structure of the attractor on the other hand. Equation (49) enhances this aspect, but there might be more general relations that we are currently researching.[39]

2.6. Finite-Size Scaling

We would now like to address the question of finite-size scaling on the basis of the results obtained above. Let us first discuss the scaling of $\mathscr{F}_L(\eta)$ (Eq. (54)). This function plays a similar role as the generating functional in quantum field theory and η acts as a conjugated field to the observable n. It must, in particular, be emphasized that the topological pressure in Eq. (54) contains *all information* about the scaling properties of the probability distribution on the space of trajectories of the avalanche observable n. Consequently, a scaling theory appears to be possible, in analogy to statistical mechanics. Formally, when $h = 0$, one can associate to each local field η_i an exponent y_i and seek a scaling form for $\mathscr{F}_L(\eta)$ such as

$$\mathscr{F}_L(\eta) = L^{\beta}\mathscr{H}(L^{y_1}\eta_1,..., L^{y_n}\eta_n,...), \tag{61}$$

where $\mathscr{H}$ is a universal function. This provides the scaling of the n points cumulants. However, one has introduced an infinite number of fields and this formula raises the question: what are the relevant fields (in the renormalization group sense, i.e., unstable directions for the renormalization flow) or in other words which scaling exponents define the universality classes in SOC? In classical critical phenomena, the renormalization group approach and the renormalization properties of the Hamiltonian and of the free energy[8] allow one to select the relevant fields and two scaling exponents are extracted: α (the specific heat exponent) and η (the correlation length). In the SOC case, the main difficulty is to adapt the renormalization schemes and to select the relevant fields in order to build the scaling theory (see ref. 36).

For the generating function (56) of the marginal probability distribution $P_L(n)$, the scaling form (61) reduces to

$$G_L(t) = L^{\beta_n(1-\tau_n)}G(tL^{\beta_n}), \tag{62}$$

where G is a universal function. This form corresponds to the Kadanoff *et al.* finite-size scaling Ansatz.[13] In particular, the cumulant $C_L(q)$ has a scaling factor:

$$\sigma(q) \stackrel{\text{def}}{=} \lim_{L\to\infty} \frac{\log(C_L(q))}{\log(L)}, \tag{63}$$

which is a linear function[22] of q, and $\sigma(q) = \beta_n(q+1-\tau_n)$ for $q \geqslant \tau_n - 1$.

Note that the measured exponents $\tau_n \in]1, 2[$ for the usual avalanche observables. Consequently, $\sigma(q) > 0$ for $q \geqslant \tau_n - 1$, and the moments of order $q \geqslant \tau_n - 1$ diverge in the limit $L \to \infty$. This implies a *loss of analyticity*

[22] Recently, it has been argued that the finite-size form (Eq. (62)) might be violated in several models such as the BTW model[14] or the Zhang model.[15] Alternative scalings where $\sigma(q)$ is a nonlinear function of q have been proposed.[14] The conclusions in refs. 14 and 15 were, however, essentially supported by numerical simulations and theoretical results are still missing. In particular, the numerical bias induced by numerics was not discussed.[24] Such a scaling entails very singular properties for the asymptotic distribution function (and for the asymptotic topological pressure, if it exists) and, consequently, for the asymptotic dynamical system. This opens up interesting questions and prospects that will be discussed in a forthcoming paper.

of the limiting generating function. As discussed below, this effect is manifested by a *Lee–Yang phenomenon*. In particular, the properties of the critical zeros are directly related to the $\sigma(q)$'s. In the case of Eq. (62), the distribution of zeros is characterized by the exponents β_n and τ_n, but relations linking the Lee–Yang zeros distribution to the $\sigma(q)$'s can be derived for more general scaling.[24]

As discussed in the previous section, there exist close relations between the fractal structure of the attractor and the avalanche distribution. This gives hints for characterizing the behavior of avalanche distributions when the size increases. A prominent example was given in ref. 18. The Ledrappier–Young formula,[50] which relates the partial Hausdorff dimension, the Lyapunov exponents and the KS entropy, allowed us to show that the *critical exponent* of avalanche size necessarily belongs to the interval $[1, 2[$ (in particular it is *strictly lower* than 2).

Other relations of this kind may be obtained from the potential (41). Let $\beta(q)$ be defined as in Eq. (42); then, from (45) we have

$$\frac{\mathcal{G}_T(\beta,\dots,\beta)}{N^{T(q-1)}} = 1. \tag{64}$$

Denote $\psi_L(\beta) = \lim_{T\to\infty} \frac{1}{T} \log \mathcal{G}_T(\beta,\dots,\beta)$; then

$$\psi_L(\beta(q)) = (q-1)\log(N). \tag{65}$$

ψ_L is directly related to the generating function of the joint probability of avalanche sizes. (In particular, were the successive avalanche sizes to be independent we would have $\psi_L(\beta(q)) = G_L(\beta(q)\log\epsilon)$.) On the other hand, $\beta(q)$ is directly connected to the structure of the attractor. Henceforth, Eq. (65) imposes constraints on the scaling of the joint probability of avalanche sizes (and henceforth on the marginal probability $P_L(s)$), related to the fractal properties of the invariant set. This has to be investigated further, and this will be done in a separate paper.

2.7. The Limit $N \to \infty$ and the Lee–Yang Phenomenon

Rather than attempting to give a definition of the thermodynamic limit for the Gibbs state, one can, as usual in equilibrium statistical mechanics of phase transitions, focus on the analyticity properties of the thermodynamic potential (free energy). More precisely, from Eq. (54), the topological pressure is nothing but the generating function for the joint distribution of the avalanche observable n. This quantity certainly exists for finite L and has to be defined also in the thermodynamic limit if we admit that there exists a limit probability when $L \to \infty$. However, if a critical state is indeed achieved, the finite-size topological pressure should develop singularities as $L \to \infty$, $h = 0$.

The topological pressure (54) is a complicated object to handle, even numerically. However, one can argue that if the generating function, (55), of the *marginal* distribution develops singularities in the thermodynamic limit, then $\mathscr{F}$ has singularities as well. In equilibrium statistical mechanics, a standard way of handling the singularities of the free energy and the scaling as $L \to \infty$ is the study of the Lee–Yang zeros.[6] In many examples, the partition function of finite-size systems is a polynomial in a variable z which typically depends on control parameters like the temperature or the external field. Since all coefficients are positive there is no zero on the positive real axis. However, in the thermodynamic limit, at the critical point, some zeros pinch the real axis at $z = 1$, leading to a singularity in the free energy. The finite-size scaling properties of the leading zeros and of the density of zeros near $z = 1$ determine the order of the transition[11] and also the critical exponents in the case of a second-order phase transition.[12] In this paper, we show numerically that the same effect arises for the generating function of $P_L(s, h)$ for $h = 0$, while there is no Lee–Yang phenomenon for $h > 0$. This property is not specific to the observable s or to the Zhang model. Indeed, we showed in ref. 24 that this property arises as soon as a probability distribution $P_L(n), n = 1 \cdots \xi_L^n < \infty$, converges to a power law $\frac{K}{n^\tau}$, $n = 1 \cdots \infty$ when $L \to \infty$. Furthermore, when $1 < \tau < 2$ (which is the case for the usual avalanche observables and all SOC models), interesting anomalous finite-size scaling effects are observed.

Since $\xi_L^n(h)$ is finite for finite L, the generating function

$$\mathscr{Z}_L(z) = \sum_{n=0}^{\xi_L^n(h)} z^n P_L(n, h), \tag{66}$$

where $z \in \mathbb{C}$, is a polynomial of degree $\xi_L^n(h)$. In particular, since $\mathscr{Z}_L(z)$ is an analytic function of z in the complex plane, all its moments exist. Since all $P_L(n, h)$ are positive, $\mathscr{Z}_L(z)$ has no zero on the positive real axis for finite L. Consequently the log-generating function $\log(\mathscr{Z}_L(z))$ is well defined on $\mathbb{R}_+^*$. More precisely, $\mathscr{Z}_L(z) > 0$ for z in a small neighborhood of $\mathbb{R}_+^*$ where $G_L(t) = \log(\mathscr{Z}_L(e^t))$ is an analytic function of z (or t).

For finite L, $\mathscr{Z}_L(z)$ has $\xi_L^n(h) + 1$ zeros in $\mathbb{C}$, which are either real $\leqslant 0$ or complex conjugate. Denote them by $z_L(k)$, $k = 0 \cdots \xi_L^n$, and order them such that $0 < |z_L(1) - 1| \leqslant \cdots \leqslant |z_L(k) - 1| \leqslant \cdots \leqslant |z_L(\xi_L^n + 1) - 1|$. Write $z_L(k) = R_L(k)\, e^{i\theta_L(k)} = e^{t_L(k)}$.

We now report the following numerical observations for the avalanche size distribution. A general theory and analytical results, extending to other models of SOC, are developed in ref. 24.

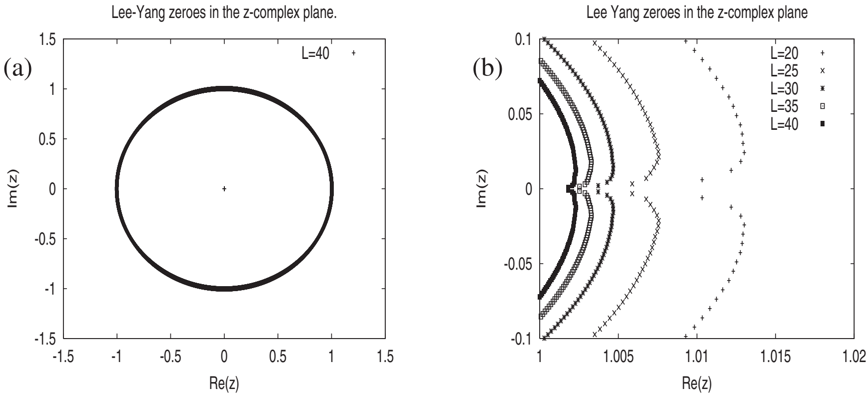

Fig. 7. Lee–Yang zeros in the z plane. (a) The curve $\mathscr{C}_L$ for $L = 40$. (b) Local behavior near $z = 1$ of the Lee–Yang zeros for various L values in the z complex plane $E_c = 2.2$, $\epsilon = 0.1$, and $h = 0$.

• The zeros lie on a curve $\mathscr{C}_L$ in the complex plane which accumulate to the unit circle. This curve, however, is *not a circle*. In particular, it has a "cusp shape" in the neighborhood of $z = 1$ (Fig. 7(a)). An analytic form is given in ref. 24.

• For $h = 0$ infinitely many zeros accumulate on $z = 1$ (see Fig. 7(b) and Fig. 8). Consequently, the pressure *ceases to be analytic in the thermodynamic limit*, as one would expect. Note that this property is equivalent to the divergence of the moments $m_L(q)$ since it can be proved that the

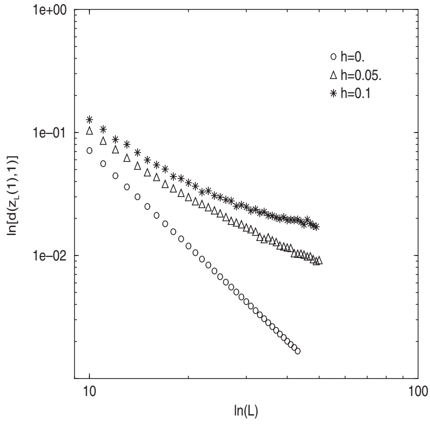

Fig. 8. Distance of the first Lee–Yang zeros $z_L(1)$ to $z = 1$ versus L, $E_c = 2.2$, $\epsilon = 0.1$ in the critical case $h = 0$ and subcritical case ($h = 0.1$, $h = 0.05$).

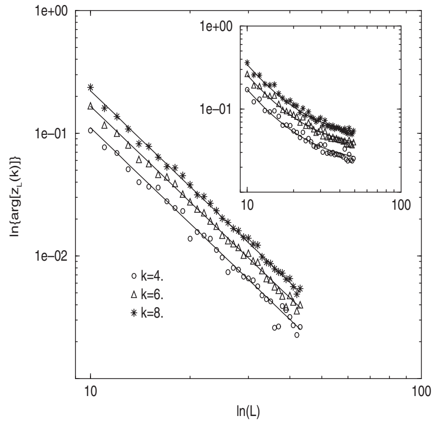

Fig. 9. Argument of the Lee–Yang zeros $z_L(4)$, $z_L(6)$, $z_L(8)$ for $E_c = 2.2$, $\epsilon = 0.1$ in the critical case $h = 0$ and in the subcritical case $h = 0.1$ (inset). In solid lines are drawn the interpolation curves $y = 2\pi k(cx^{-2})^{\frac{1}{\sigma}}$ (conservative case) and $y = 2\pi k(a + cx^{-2})^{\frac{1}{\sigma}}$ (nonconservative case) where a and c have been determined by regression.

moments $m_L(q)$, for any q larger than some q_0 diverge if and only if a Lee–Yang phenomenon occurs. This opens a way toward a scaling theory from the behavior of the Lee–Yang zeros, in a way similar to what was done in equilibrium statistical mechanics.[7, 9–12]

• For $h > 0$ (subcritical regime) the zeros do not pinch $z = 1$. This is a clear indication that in this case there is no critical state in the thermodynamic limit (Fig. 8).

• There exist characteristic features of the zeros that can be connected to the critical behavior.

– For $h = 0$, the angle $\theta_L(k)$ (argument of $z_L(k)$) scales like $\theta_L(k) \sim L^{\beta}$ (Fig. 9). It can be proved that, for a probability distribution converging to a power law and obeying a finite-size scaling form $s^{-\tau} g(\frac{s}{L^{\beta}})$, this scaling is exact (with, however, a slight deviation for the first two zeros[24]). More generally, if the probability distribution obeys the scaling form (29), then one can prove that the angle $\theta_L(k)$ scales like:

$$\theta_L(k) \sim \frac{2\pi k}{\xi_L^n(h)}. \tag{67}$$

For $n = s$, the avalanche size, it follows from Eqs. (26) and (30) that $\theta_L(k) \sim \frac{2\pi k}{\bar{e}_L^{-1/\sigma}}$, namely that

$$\theta_L(k) \sim \frac{2\pi k}{[a + cL^{-2}]^{-\frac{1}{\sigma}}}, \tag{68}$$

where a and c are proportional to $2(1-\gamma)\,\epsilon\bar{X}_L^+$ and $\bar{X}_L^+$ (see Eq. (26)) respectively. a and c depend slightly on L (via $\bar{X}_L^+$) but they converge rapidly to a constant as $L \to \infty$ (see Fig. 4(a) for the L dependence of $\bar{X}_L^+$). Therefore, studying the L dependence of $\theta_L(k)$ gives a straightforward way of computing σ (or $\beta = \frac{2}{\sigma}$). We performed a numerical study of the angles $\theta_L(k)$ for $E_c = 2.2$, $\epsilon = 0.1$ in the conservative and nonconservative case ($h = 0.1$) for $L = 10$ to $L = 50$. We obtained $\beta = 2.59 \pm 0.04$ and $\sigma = 0.772$. This corresponds to a critical exponent $\tau = 1.227$, not so far away from the theoretical value $\tau = 1.253$ obtained by the renormalization group analysis.[37] Going to a larger size would certainly improve the accuracy, though one has to be careful of the pathological bias induced by the standard numerical procedure, where the number of samples is fixed independently of the system size.[24] In Fig. 9, we plotted the angle $\theta_L(k)$ for $k = 4, 6, 8$ and a nonlinear fit where the constants a and c in Eq. (68) were used as fitting parameters.

– In many models of statistical mechanics the singular part of the free energy obeys a finite-size scaling form $f^s(t, V) = \frac{1}{V} W[t(AV)^{\frac{1}{2-\alpha}}]$ (equivalent to Eq. (62)),[8–10] and the first zeros in the t plane ($t = \log(z)$) form a characteristic angle with the real axis, independent of L, which can be related to the specific heat exponent α. Here the angle in the t plane *violates the usual scaling and depends weakly on* L (in fact, the violation is logarithmic[24]). It can be proved that this effect *is not an indication that the topological pressure does not obey finite-size scaling* but is simply due to the value of the exponent $\tau > 1$.

3. DISCUSSION AND CONCLUSION

In this paper we have discussed the application of a thermodynamic formalism to the Zhang model of Self-Organized Criticality. We have shown that under physically natural assumptions Gibbs measures can be defined to characterize various statistical properties of the finite-size SOC state. This opens up the possibility of building the equivalent of a statistical mechanics theory of critical phenomena for SOC models. It might also open the way toward a general setting in which concepts like universality classes could be properly defined. Indeed, although the extrapolation of the thermodynamic formalism to the infinite lattice size limit needs further developments, this work suggests that the topological pressure can be used as an indicator of a phase transition. In particular, we found a Lee–Yang phenomenon for a quantity derived from the pressure, and we noticed that several characteristic patterns emerge, which could allow for a future classification of the models.

We would now like to discuss some points that have not been developed in the paper.

1. *Extensions of the Thermodynamic Formalism.* In this paper, we have focused on the most common potentials which are directly related to dynamical properties and to the fractal structure of the support of the invariant measure. We have also shown that they are related to the avalanche-size distribution. We now intend to construct more general potentials allowing, on one hand, an investigation of the properties of other avalanche observables. On the other hand, we showed numerically in ref. 19 that the Lyapunov spectrum exhibits a finite-size scaling property with a universal exponent τ_λ related to the anomalous diffusion exponent. It would be worth showing this property analytically by producing a suitable potential.

2. *Spectral-Gap Vanishing.* In the finite system, the exponential correlation decay along the time trajectory is given by the spectral gap between the largest eigenvalue of the Markov matrix $\mathscr{W}$ (which is 1) and the second eigenvalue. As mentioned above, this gap is positive whenever $\mathscr{A}$ is mixing. When L diverges, $\mathscr{N}_L$ diverges for $h=0$ since $\mathscr{N}_L \geqslant \xi_L^s$. On the other hand, the notion of critical phenomena involves a non-exponential correlation decay or a divergence of the time-correlation length, which is the inverse of the spectral gap. Consequently, one expects that for $h=0$ the spectral gap vanishes in the thermodynamic limit. It might be useful to compute the spectral gap and, in particular, its L dependence. This could be achieved from standard techniques on Markov chains,[43] provided we have additional information about the structure of $\mathscr{W}$. From the analogy with critical phenomena we expect the gap to vanish like L^η, where η plays the role of the exponent giving the spatial correlation decay in statistical mechanics. This could be a new exponent that might be related to the exponent τ_λ which we have found in ref. 19.

3. *Explicit Form of the Energy Density.* In Section 2.4 we used the result $\mu_L\{\mathscr{U}_\eta(\mathscr{S})\} \sim \eta^\alpha$, for which we only offered a numerical test. An analytic computation would certainly be more satisfactory. This could be done if we could compute the energy profile depicted in Fig. 5(b) This would also allow us to elucidate the particular shape with several peaks, which is still an open problem in SOC.

These points are under current investigation.

APPENDIX

In this appendix, devoted to non-specialists, we give a brief summary of the thermodynamic formalism used in Section 2.5 to construct Gibbs measures. Useful references are refs. 22, 23, 27, and 48.

Assume that we have a set of symbols $\omega \in \mathscr{I}$ and a transition matrix $\mathscr{A}$. Call $\mathscr{X}$ ($\mathscr{X}^+$) the set of bi-infinite (resp. right infinite) legal sequences. Write $[\tilde{\omega}]_n$ for an n-cylinder (this is the subset of $\mathscr{X}$ where the sequences have the same n first digits as $\tilde{\omega}$). Denote by $\sigma_{\mathscr{A}}$ the shift on $\mathscr{X}$. In the following we restrict our discussion to the set of right infinite sequence $\mathscr{X}^+$, as is common in the framework of the thermodynamic formalism.[27, 48] There exists a formal analogy between a sequence of $\mathscr{X}^+$ and a (right-infinite) one-dimensional chain of Potts-like spins taking values in $\mathscr{I}$. The transition matrix $\mathscr{A}$ then acts as a hard-core-like potential in the sense that, if the spin at the tth place in the chain has a value ω_t, the next spin (at the place $t+1$) can only take values in the subset of $\mathscr{I}$ such that $\mathscr{A}_{\omega_t, \omega_{t+1}} = 1$. Other transitions are forbidden. This formalism allows, in particular, a study of a class of dynamically relevant invariant measures of the dynamical system as formal analogy to Gibbs measure in a chain of spins.

Let $F(\mathscr{X}^+)$ be the space of Hölder continuous functions for a metric $d_\theta(x, y) = \theta^N$ on $\mathscr{X}^+$, where N is the largest nonnegative integer such that $x_i = y_i$, $i < N$, and $0 < \theta < 1$.[48, 27] We call a *potential* an element of $F(\mathscr{X}^+)$. In particular, a potential has the following property:

$$\mathrm{var}_n(\phi) \leqslant C\theta^n, \qquad n \geqslant 0, \tag{69}$$

for some $C > 0$ and some $\theta \in]0, 1[$, where $\mathrm{var}_n(\phi) = \sup\{|\phi(x) - \phi(y)|, x_i = y_i, i < n\}$. Note that Eq. (69) is the equivalent of the exponential decrease of the interaction with distance, ensuring the existence of a thermodynamic limit in statistical mechanics.[4, 5] A *finite-range* potential of order r is such that $\phi(x) = \phi(y)$ if $x_i = y_i$ for $0 \leqslant i < r$; namely the values that ϕ takes depend only on the first symbols. Any infinite-range potential in $F_\theta(\mathscr{X}^+)$ can be uniformly approximated by a sequence of r-range potentials.

Set $S_T\phi(\tilde{\omega}) = \sum_{t=1}^{T} \phi(\sigma_{\mathscr{A}}^t \tilde{\omega})$, where ϕ is a potential. Define the finite *partition function* by

$$Z_T(\phi) = \sum_{\tilde{\omega} \in \chi_T^+} \exp\{S_T\phi(\tilde{\omega})\}, \tag{70}$$

and the *pressure* (the free energy density) by

$$\mathscr{F}(\phi) = \lim_{T \to \infty} \frac{1}{T} \log Z_T(\phi). \tag{71}$$

It can be proved that this limit exists provided ϕ decays sufficiently fast (see Eq. (69)), namely there is no "phase transition" provided the potential belongs to $F(\mathscr{X}^+)$.

A *Gibbs measure* is an invariant measure where the potential gives an exponential weight to the cylinders. More precisely, μ_ϕ is a Gibbs measure if there exists an A such that for all $T > 0$ and $\omega \in \chi_T^+$

$$A^{-1} \leqslant \frac{\mu_\phi([\tilde{\omega}]_T)}{e^{S_T \phi(\tilde{\omega}) - T\mathscr{F}(\phi)}} \leqslant A. \tag{72}$$

This essentially means that μ_ϕ is exponential with a weight given by the sum of the values that ϕ takes on the orbit of ω. Since $Z_T(\phi) \sim e^{T\mathscr{F}(\phi)}$, the measure of a spin chain of length T is $\sim \frac{\exp(\sum_{t=1}^{T} \phi(\sigma^t \tilde{\omega}))}{Z_T(\phi)}$ and the formal analogy with statistical mechanics is straightforward.

In this setting one also associates to each potential ϕ the *Ruelle operator* $\mathscr{L}_\phi$, which is a formal extension of the Kramers–Wannier transfer matrix for a spin chain.[4] An extension of the Perron–Frobenius theorem for matrices, due to Ruelle, shows that when $\mathscr{A}$ is irreducible and aperiodic, $\mathscr{L}_\phi$ admits a unique maximal positive eigenvalue which is equal to $e^{\mathscr{F}(\phi)}$. The corresponding left eigenvector is the Gibbs measure μ_ϕ. The spectrum of $\mathscr{L}_\phi$ provides information about the (strong) mixing properties of μ_ϕ. In particular, the spectral gap between the largest eigenvalue ($e^{\mathscr{F}(\phi)}$) and the remaining part of the spectrum determines the dominant exponential decay rate of the correlation functions (or decay rate to equilibrium).

μ_ϕ also satisfies a *variational principle* analogous to the free-energy minimization in statistical mechanics. Call $h(\mu)$ the entropy of the invariant measure μ; then the quantity $h(\mu) + \int \phi \, d\mu$ admits a unique maximum for $\mu = \mu_\phi$ equal to the pressure $\mathscr{F}(\phi)$. The maximizing measure is naturally called an *equilibrium state*. Each equilibrium state for a potential $\phi \in F(\mathscr{X}^+)$ is a Gibbs state. There exists only one maximum (or one equilibrium state related to the observable ϕ) when $\mathscr{A}$ is mixing. This situation corresponds to the absence of a phase transition in statistical mechanics.

The pressure, beyond the variational principle, shares others characteristics with the free energy: it is convex, nondecreasing, sub-additive ($\mathscr{F}(\phi_1 + \phi_2) \leqslant \mathscr{F}(\phi_1) + \mathscr{F}(\phi_2)$) and is a generating function for the expectations with respect to μ_ϕ. More precisely, the following can be proved.[27, 48] Let $\phi, \eta \in F_\theta$, and set $\mathscr{P}(t) \stackrel{\text{def}}{=} \mathscr{F}(\phi + t\eta)$, where $t \in \mathbb{R}$; then

$$\mathscr{P}'(0) = \frac{d}{dt} \mathscr{F}(\phi + t\eta)\bigg|_{t=0} = \int \eta \, d\mu_\phi \stackrel{\text{def}}{=} E[\eta]_\phi, \tag{73}$$

where $E[\,]_\phi$ is the expectation with respect to μ_ϕ. In the same way:

$$\sigma_\phi^2(\eta) = \mathscr{P}''(0) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E[\eta(t)\, \eta(0)]_\phi - E[\eta]_\phi^2, \tag{74}$$

where $E[\eta(t)\,\eta(0)]_\phi - E[\eta]_\phi^2$ is the correlation function of the function η at time t and where the average is performed with respect to μ_ϕ. This can be generalized to correlation functions between different observables and to higher order.[45] The coefficient $\sigma_\phi(\eta)$ characterizes the average fluctuations of η along trajectories weighted by the measure μ_ϕ. Provided that $\sigma_\phi(\eta) > 0$ the central limit theorem holds, namely the fluctuations are Gaussian (more precisely $\lim_{T\to\infty} \text{Prob}[\frac{\sum_{t=1}^{T}\eta(t) - TE[\eta]_\phi}{\sigma_\phi(\eta)\sqrt{T}} < y] = \mathcal{N}(y)$, where $\mathcal{N}$ is the characteristic function of the Gaussian distribution).

From the relation (74) one can extract Green–Kubo transport coefficients from microscopic quantities.[44, 45]

According to the choice of potential one is able to extract different information about the statistical properties of the dynamics. The situation is analogous to that encountered in statistical mechanics, where the choice of the thermodynamic potential corresponds to a different choice of *ensemble*. However, one has *a priori* an infinite number of choices for the potential (resp. measure), but only a few of them are physically significant.

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REFERENCES

1. P. Bak, C. Tang, and K. Wiesenfeld, Self organized criticality: An explanation of $1/f$ noise, *Phys. Rev. Lett.* **59**:381–384 (1987); Self organized criticality, *Phys. Rev. A*. **38**: 364–374 (1988).
2. P. Bak, *How Nature Works* (Springer-Verlag, 1996).
3. H. J. Jensen, Self-organized criticality: Emergent complex behavior in physical and biological systems, in *Cambridge Lecture Notes in Physics*, Vol. 10 (Cambridge University Press, 1998).
4. D. H. Meyer, The Ruelle–Araki transfer operator in classical statistical mechanics, *Lecture Notes in Physik*, Vol. 123 (Springer-Verlag).
5. D. Ruelle, *Statistical Mechanics: Rigorous Results* (Benjamin, New York, 1969).
6. C. N. Yang and T. D. Lee, Statistical theory of equations of state and phase transitions I. Theory of condensation, *Phys. Rev.* **87**:404–409 (1952); Statistical theory of equations of state and phase transitions II. Lattice gas and Ising model, *Phys. Rev.* **87**:410–419 (1952).
7. R. Abe, Note on the critical behavior of Ising ferromagnets, *Prog. Theor. Phys.* **38** (1967)
8. M. J. Fisher, The renormalization group in the theory of critical behaviour, *Rev. Mod. Phys.* **46**:597–616 (1974).
9. C. Itzykson, R. B. Pearson, and J. B. Zuber, Distribution of zeros in Ising and Gauge models, *Nuclear Phys. B* **220[FS8]**:415–433 (1983).

10. M. L. Glasser, V. Privman, and L. S. Schulman, Complex-temperature-plane zeros: Scaling theory and multicritical mean-field models, *Phys. Rev. B* **33** (1987).
11. W. Janke and R. Kenna, The strength of first and second order phase transitions from partition function zeros, *J. Stat. Phys.* **102**:1221–1227 (2001).
12. R. J. Creswick and S. Y. Kim, Finite-size scaling of the density of zeros of the partition function in first and second order transitions, *Phys. Rev. E* **56**:2418 (1997).
13. L. P. Kadanoff, S. R. Nagel, L. Wu, and S. Zhou, Scaling and universality in avalanches, *Phys. Rev. A* **39**:6524–6527 (1989).
14. C. Tebaldi, M. De Menech, and A. Stella, *Phys. Rev. Lett.* **83**:3952 (1999).
15. R. Pastor-Satorras and A. Vespignani, Anomalous scaling in the Zhang model, *Eur. Phys. J. B* **18**:197–200 (2000)
16. Ph. Blanchard, B. Cessac, and T. Krüger, A dynamical systems approach for SOC models of Zhang type, *J. Stat. Phys.* **88**:307–318 (1997).
17. B. Cessac, Ph. Blanchard, and T. Krueger, A dynamical system approach to self-organized criticality, in *Mathematical Results in Statistical Mechanics*, Marseille 1998 (Word Scientific, Singapore).
18. Ph. Blanchard, B. Cessac, and T. Krüger, What can one learn about self-organized criticality from dynamical system theory ?, *J. Stat. Phys.* **98**:375–404 (2000).
19. B. Cessac, Ph. Blanchard, and T. Krüger, Lyapunov exponents and transport in the Zhang model of self-organized criticality, *Phys. Rev. E* **64**:016133 (2001).
20. H. Y. Zhang, Scaling theory of self-organized criticality, *Phys. Rev. Lett.* **63**:470–473 (1988).
21. Ya. G. Sinai, Gibbs measures in ergodic theory, *Russ. Math. Surveys* **27**:21–69 (1972)
22. D. Ruelle, *Thermodynamic Formalism* (Addison–Wesley, Reading, MA, 1978).
23. R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, *Lect. Notes in Math.*, Vol. 470 (Springer-Verlag, Berlin, 1975).
24. B. Cessac and J. L. Meunier, Anomalous scaling and Lee–Yang zeroes in self-organized criticality, *Phys. Rev. E* **65**:1–18 (2002).
25. D. Dhar, *Phys. Rev. Lett.* **64**:1613 (1990); D. Dhar and S. N. Majumdar, *J. Phys. A* **23**:4333 (1990); S. N. Majumdar and D. Dhar, *Physica A* **185**:129 (1992); D. Dhar and R. Ramaswamy, *Phys. Rev. Lett.* **63**:1659 (1989).
26. D. Sornette, A. Johansen, and I. Dornic, Mapping self-organized criticality onto criticality, *J. Phys. France* **5**:325–335 (1995).
27. G. Keller, *Equilibrium States in Ergodic Theory* (Cambridge University Press, 1998).
28. A. Katok and B. Hasselblatt, *Introduction to Dynamical Systems Theory* (Kluwer, 1996).
29. A. Vespignani and S. Zapperi, How self-organized criticality works: A unified mean-field picture, *Phys. Rev. E* **57**:6345 (1998).
30. V. I. Oseledec, A multiplicative ergodic theorem. Characteristic Lyapunov exponents of dynamical systems, *Trans. Moscow Math. Soc.* **19**:179–210 (1968).
31. M. Pollicott, Lectures on ergodic theory and Pesin theory on compact manifolds, *London Math. Soc., Lect. Notes Series*, Vol. 180 (Cambridge University Press, 1993).
32. J. P. Eckmann and D. Ruelle, Ergodic theory of strange attractors, *Rev. Mod. Phys.* **57**:617 (1985); J. P. Eckmann, O. Kamphorst, D. Ruelle, and S. Cilliberto, *Phys. Rev. A* **34**:4971 (1986).
33. Eurandom conference, *Longe Range Stochastic Dynamics*, Eindhoven, December 2001.
34. Ph. Flajolet, *Theoretical Computer Sci.* **215**:371–381 (1999).
35. E. Seneta, *Non-Negative Matrices* (Allen and Unwin, London, 1973).
36. D. Volchenkov, Ph. Blanchard, and B. Cessac, Quantum field theory renormalization group approach to self-organized criticality: The case of random boundaries, *Int. J. Mod. Phys. B* **16**:1171–1204 (2002).

37. L. Pietronero, A. Vespignani, and S. Zapperi, *Phys. Rev. Lett.* **72**:1690 (1994); A. Vespignani, S. Zapperi, and L. Pietronero, *Phys. Rev. E.* **51**:1711 (1995).
38. K. Falconer, *Techniques in Fractal Geometry* (Wiley, 1997).
39. Ph. Blanchard, B. Cessac, and S. Sequeira, in preparation.
40. K. J. Falconer, Bounded distortion and dimension for non-conformal repellers, *Math. Proc. Camb. Phil. Soc.* **115**:315–334 (1994).
41. L. M. Barreira, A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems, *Ergod. Theor. Dynam. Syst.* **16**:871–927 (1998).
42. K. J. Falconer, Generalized dimensions of measures on self-affine sets, *Nonlinearity* **12**:877–891 (1999).
43. M. Benaim and N. E. Karoui, Promenades aléatoires, Cours de l'école polytechnique.
44. B. Dorfmann, *An Introduction to Chaos in Nonequilibrium Statistical Mechanics* (Cambridge University Press, 1999).
45. P. Gaspard, Chaos, scattering, and statistical mechanics, in *Cambridge Nonlinear Science*, Series 9 (Cambridge University Press, Cambridge, 1998).
46. C. Beck and F. Schloegl, Thermodynamics of chaotic systems, in *Cambridge Nonlinear Science*, Series 4 (Cambridge University Press, Cambridge, 1993).
47. P. Bak and C. Tang, Critical exponents and scaling relations for self-organized critical phenomena, *Phys. Rev. Lett.* **60**:2347–2350, (1988).
48. W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, *Collection Asterisque* (Société Mathématique de France, 1990), pp. 187–188.
49. A. Barrat, A. Vespignani, and S. Zapperi, Fluctuations and correlations in sandpile models, *Phys. Rev. Lett.* **83**:1962 (1999).
50. F. Ledrappier and L. S. Young, The metric entropy for diffeomorphisms, *Ann. Math.* **122**:509–574 (1985).
51. O. M. Sarig, Thermodynamic formalism for countable markov shifts, *Ergod. Th. Dyn. Sys.* **19**:1565–1593 (1999).
52. L. S. Young, Ergodic theory of differentiable dynamical systems.
53. C. Maes, F. Redig, E. Saada, and A. Van Moffaert, On the thermodynamic limit for a one-dimensional sandpile process, *Markov Process. Related Fields* **6**:1–21 (2000).